MATRIX THEORY AND THE WORLDSHEET OF THE DISCRETE LIGHT-FRONT SUPERSTRING

GORDON W. SEMENOFF

UNIVERSITY OF BRITISH COLUMBIA

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G. GRIGNANI, P. ORLAND, L. PANIAK & G.W.S.

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CONTENTS:

1) LIGHT-LIKE COMPACTIFICATION OF STRING THEORY

2) A THEOREM ABOUT BRANCHED COVERS

3) IMPLICATIONS FOR MATRIX THEORY.

SUMMARY AND CONCLUSIONS

CONSIDER PATH INTEGRAL FOR STRING WITH

COMPACTIFICATION OF NULL DIRECTION $X^+ \sim X^+ + 2\pi R$

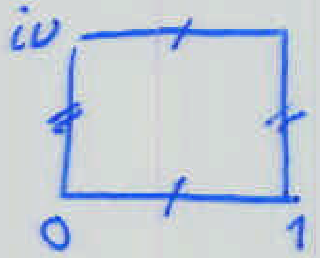
FINITE TEMPERATURE $X^0 \sim X^0 + i\beta \qquad \beta = 1/T$

1) WORLDSHEETS WHICH CONTRIBUTE TO PATH INTEGRAL ARE GENUS g RIEMANN SURFACES, Σ_g, CONSTRAINED BY

$$\sum_{i=1}^{g} (m_i + i\nu n_i)\,\Omega_{ij} = (r_j + i\nu s_j) \qquad \nu = \frac{\sqrt{2}\beta R}{4\pi\alpha'}$$

2) Ω_{ij} OBEYS THIS CONSTRAINT <u>IF AND ONLY IF</u>

Σ_g IS A BRANCHED COVER OF T^2



3) IN MATRIX STRING THEORY (MAXIMALLY SUPERSYMMETRIC 1+1-DIM YANG-MILLS THEORY)

$$p^+ = RH \qquad p^- = N/R \qquad p^0 = \frac{1}{\sqrt{2}}(p^+ + p^-)$$

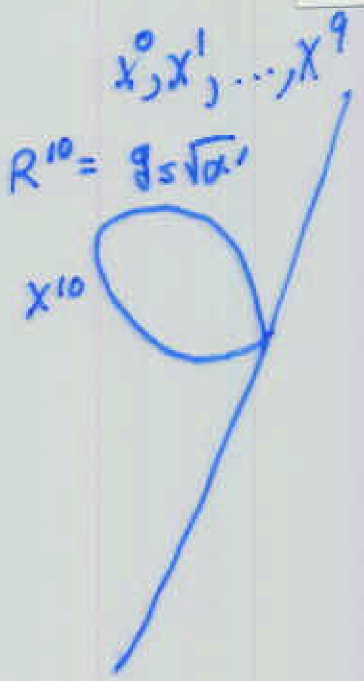
$$Z = \sum_{N=0}^{\infty} e^{-\frac{\beta N}{\sqrt{2}R}}\, \mathrm{TR}\, e^{-\frac{\beta R}{\sqrt{2}} H}$$

$g_s \mapsto 0 \sim g_{YM} \to \infty$

SIMULTANEOUS EIGENVALUES OF $X^I(\sigma_1,\sigma_2)$ BECOME STRING COORDINATES $\det\left(X^I(\sigma_1,\sigma_2) - x^I(\sigma_1,\sigma_2)\right) = 0$

$x^I(\sigma_1,\sigma_2)$ LIVES ON BRANCHED COVERS OF T^2.

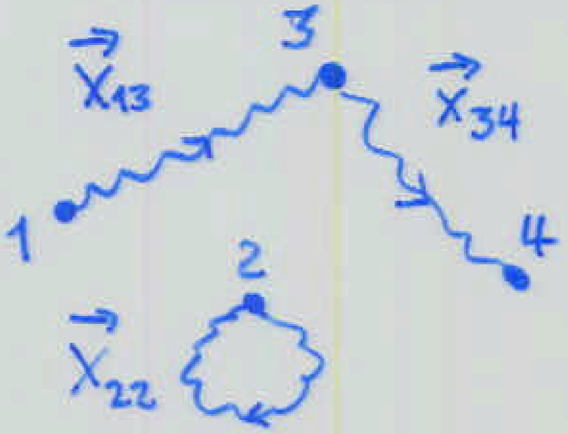
MATRIX MODEL OF M-THEORY



IIA SUPERSTRING =

M-THEORY COMPACTIFIED ON A CIRCLE $x^{10} \sim x^{10} + 2\pi g_s \sqrt{\alpha'}$

UNITS OF KK-MOMENTUM = # OF D0-BRANES, $P^{10} = N/R^{10}$



{ LOW ENERGY DYNAMICS
0+1-DIM SUPER YANG-MILLS TH.
GAUGE GROUP U(N) }

$$S = \int dt \, \frac{1}{g_{YM}^2} \, \mathrm{TR} \left\{ (DX^I)^2 - \sum_{I<J} [X^I, X^J]^2 + i\psi^\dagger D\psi - \bar{\Psi}\Gamma^I [X^I, \psi] \right\}$$

⇓

MATRIX MODEL DESCRIBES M-THEORY COMPACTIFIED ON A LIGHT-LIKE CIRCLE OF RADIUS R

$$x^+ \sim x^+ + 2\pi R$$

$$P^+ = RH \qquad P^- = N/R$$

IIA STRING WITH COMPACT NULL DIRECTION $x^+ \sim x^+ + 2\pi R$ AND $P^- = N/R$ OBTAINED BY COMPACTIFYING

$$x^9 \sim x^9 + 2\pi g_s \sqrt{\alpha'}$$

DOES THIS PRODUCE PERTURBATIVE STRING THEORY?

The free energy of the superstring is given by the path integral

$$\beta F V = -\sum_{g,\sigma} g_s^{2g-2}.$$

$$\cdot \int [dh_g dX^\mu d\psi^\mu] e^{-\frac{1}{4\pi\alpha'}\int \sqrt{|h|}(\partial X h^{-1}\partial X - 2\pi\alpha' i\psi\gamma\cdot\partial\psi)}$$

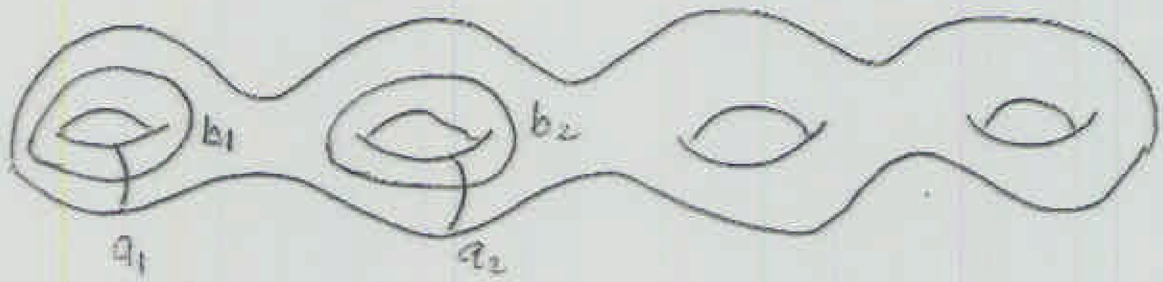
Compactify Euclidean time

$$X^0 \sim X^0 + \beta$$

Compactification of the light cone,

$$(X^0, X^9) \sim (X^0 + \sqrt{2}\pi R i, X^9 + \sqrt{2}\pi R)$$

Consider a Riemann surface Σ_g of genus g.



Homology basis of cycles $a_1, \ldots, a_g; b_1, \ldots b_g$ where

$$a_i \cap a_j = \emptyset \qquad b_i \cap b_j = \emptyset$$

$$a_i \cap b_j = \delta_{ij}$$

There are g linearly independent holomorphic 1-forms $\omega_i, \ldots, \omega_g$ A basis can be chosen so that

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \oint_{b_i} \omega_j = \Omega_{ij}$$

Period matrix

$\Omega_{ij} = \Omega_{ji}$, $\Omega = \Omega_1 + i\Omega_2$, $\Omega_2 > 0$.

Anti-holomorphic differentials, $\bar{\omega}$ have the same properties with $\Omega \to \Omega^*$.

First homology group $H_1(\Sigma_g, C)$ is a complex vector space spanned by (a_i, b_i).

First cohomology group $H^1(\Sigma_g, C)$ is the dual vector space spanned by $(\omega_i, \bar{\omega}_i)$

The inner product

$$(\lambda_i \omega_i + \xi_i \bar{\omega}_i) \cdot \left(\delta_j a_j + \gamma_j b_j\right) =$$

$$= (\lambda + \xi) \cdot (\delta + \Omega_1 \gamma) + i(\lambda - \xi)\Omega_2 \cdot \gamma$$

follows from

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \oint_{b_i} \omega_j = \Omega_{ij}$$

$$\oint_{a_i} \bar{\omega}_j = \delta_{ij} \qquad \oint_{b_i} \bar{\omega}_j = \Omega^*_{ij}$$

Include in the path integral the sectors where the worldsheet wraps the compact dimensions

$$dX^0 = \sum_{i=1}^{g} \left(\lambda_i \omega_i + \bar{\lambda}_i \bar{\omega}_i\right) + \text{exact}$$

$$dX^9 = \sum_{i=1}^{g} (\gamma_i \omega_i + \bar{\gamma}_i \bar{\omega}_i) + \text{exact}$$

Determine coefficients so that (X^0, X^9) have correct holonomy,

$$\oint_{a_i} dX^0 = m_i \beta + p_i \sqrt{2} \pi R i$$

$$\oint_{a_i} dX^9 = p_i \sqrt{2} R$$

$$\oint_{b_i} dX^0 = n_i \beta + q_i \sqrt{2} \pi R i$$

$$\oint_{b_i} dX^9 = q_i \sqrt{2} \pi R$$

The Bosonic part of the action depends on the integers is

$$\frac{1}{4\pi\alpha'}\int dX * dX = \frac{\beta^2}{4\pi\alpha'}(n\Omega^* - m)\Omega_2^{-1}(\Omega n - m) +$$

$$+2\pi i\frac{\sqrt{2}\beta R}{4\pi\alpha'}\mathrm{Re}\left((p\Omega^* - q)\Omega_2^{-1}(\Omega n - m)\right) + ...$$

Summation over (p_i, q_i) inserts a theta function and periodic delta function into the path integral

$$\sum_{mnrs} e^{-\frac{\beta^2}{4\pi\alpha'}(n\Omega^* - m)\Omega_2^{-1}(\Omega n - m)} |\tau|^{-2g} |\det\Omega_2| \cdot$$

$$\cdot \prod_{j=1}^{g} \delta^2\left(\Sigma_{i=1}^{g}(m_i + \tau n_i)\Omega_{ij} - (r_j + \tau s_j)\right)$$

$$\tau = \frac{\sqrt{2}\beta R}{4\pi\alpha'} i$$

Theorem: Σ_g is a branched cover of a torus T^2 with Teichmüller paramter τ *if and only if*

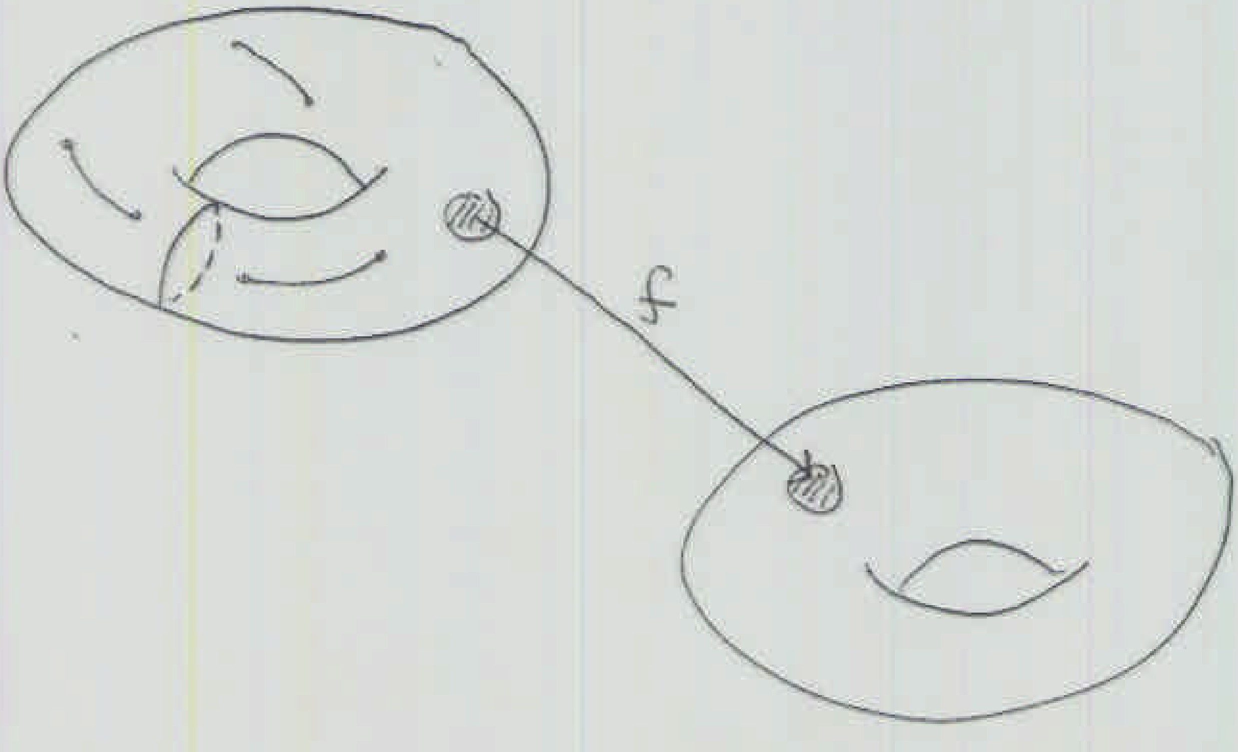
$$\sum_{i=1}^{g} (m_i + \tau n_i)\,\Omega_{ij} - \left(r_j + \tau s_j\right) = 0$$

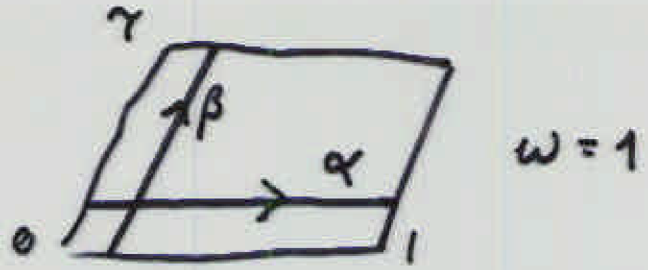
for some $m, n, r, s \in Z$,

Proof: i) Σ_g is a branched cover of the torus if there exists a covering map

$$f : \Sigma_g \to T^2$$

which is continuous, onto and holomorphic.





Consider the canonical homology basis of T^2, (α, β) and the holomorphic 1-form ω normalized so that

$$\oint_\alpha \omega = 1 \qquad \oint_\beta \omega = \tau$$

The covering map induces a homomorphism of the homology groups of Σ_g and T^2

$$f(a_i) = m_i\alpha + n_i\beta \ , \ f(b_i) = r_i\alpha + s_i\beta$$

and a linear mapping between the vector spaces

$$f : H_1(\Sigma_g; C) \to H_1(T^2; C)$$

This induces a pull-back mapping of the dual spaces

$$f^* : H^1(T^2; C) \to H^1(\Sigma_g; C)$$

with the property that $f^*(\omega)$ is a holomorphic 1-form on Σ_g (since f is a holomorphic map) and

$$\oint_{a_i} f^*(\omega) = f^*(\omega) \cdot a_i \equiv \omega \cdot f(a_i) = m_i + \tau n_i$$

$$\oint_{b_i} f^*(\omega) = f^*(\omega) \cdot b_i \equiv \omega \cdot f(b_i) = r_i + \tau s_i$$

There is a combination of cycles of Σ_g which is annihilated by any holomorphic 1-form

$$0 = \omega_k \cdot \left(\sum_j \Omega_{ij} a_j - b_i \right) = \sum_j \oint_{a_j} \omega_k \Omega_{ij} - \oint_{b_i} \omega_k$$

Applying this to $f^*(\omega)$ gives the desired identity

$$0 = f^*(\omega) \cdot \left(\sum_j \Omega_{ij} a_j - b_i \right)$$

$$= \omega \cdot \left(\sum_j \Omega_{ij} f(a_j) - f(b_i) \right)$$

$$= \sum_{i=1}^{g} (m_i + \tau n_i) \Omega_{ij} - \left(r_j + \tau s_j \right)$$

ii)Assume that the identity

$$0 = \sum_{i=1}^{g} (m_i + \tau n_i)\,\Omega_{ij} - \left(r_j + \tau s_j\right)$$

is true. We can use it to construct a covering map. Consider the following integral

$$z(P) = \int_{P_0}^{P} \sum_{i=1}^{g} \lambda_i \omega_i$$

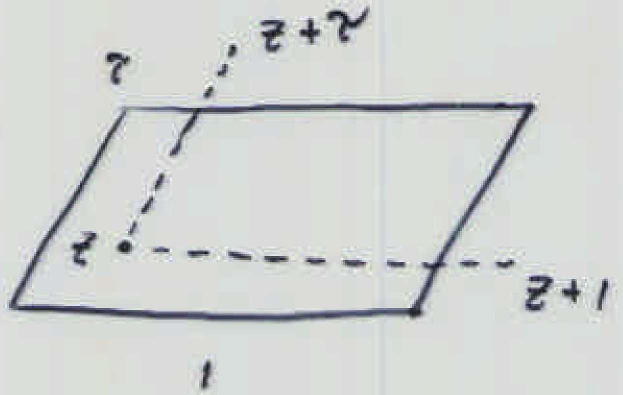
This integral maps points $P \in \Sigma_g$ to complex numbers $z(P)$. It depends on the path on which we integrate. A change of path by $t_i a_i + u_i b_i$ $(t_i, u_i \in \mathbb{Z})$ changes the r.h.s. by

$$\lambda_i \left(t_i + \left(\Omega_{ij} u_j\right)\right)$$

If we choose $\lambda_i = (m_i + \tau n_i)$, this change is

$(m_i + \tau n_i)t_i + (r_i + \tau s_i)u_i =$ integer + integer$\cdot\tau$

so $z(P) \in T^2$. This gives an explicit construction of a covering map.



Matrix string theory is a 1+1-dimensional maximally supersymetric Yang-Mills theory with gauge group $\mathrm{U}(N)$

$$S=\int d^2\sigma \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}(D_\mu\phi^I)^2-\frac{g_{\mathrm{YM}}^2}{2}\sum_{IJ}\left[\phi^I,\phi^J\right]^2 +\frac{i}{2}\bar{\psi}\Gamma_\mu D_\mu\psi-\frac{ig_{\mathrm{YM}}}{2}\bar{\psi}\Gamma_I\left[\phi^I,\psi\right]\right)$$

$\mu,\nu=1,2,\ I,J=1,\ldots,8,\ (\Gamma_\mu,\Gamma_I)$ are 16×16 10-D Dirac matrices for Weyl-Majorana fermions.

$$F_{\mu\nu}(\sigma)=\partial_\mu A_\nu(\sigma)-\partial_\nu A_\mu(\sigma)-ig_{\mathrm{YM}}[A_\mu(\sigma),A_\nu(\sigma)]$$

$$D_\mu\ldots=\partial_\mu\ldots-ig_{\mathrm{YM}}\left[A_\mu,\ldots\right]$$

Matrix string theory =IIA superstring theory with coupling constant $g_s=1/\sqrt{\alpha'}g_{\mathrm{YM}}$,

$$X^+\sim X^+ + 2\pi R$$

and N units of momentum,

$$P^-=N/R\quad,\quad P^+=HR$$

M-theory with $X^{10}\sim X^{10}+\sqrt{\alpha'}g_s$

$$\frac{X^0+X^9}{\sqrt{2}}\sim\frac{X^0+X^9}{\sqrt{2}}+2\pi R$$

$$P^0 = \frac{1}{2}\left(P^+ + P^-\right) = (N/R + H)/\sqrt{2}$$

Thermodynamic partition function is ($\beta = 1/k_B T$)

$$e^{-\beta V F[\beta]} = \mathrm{Tr} e^{-\beta P^0} = \sum_N e^{-\beta N/\sqrt{2}R} \mathrm{Tr} e^{-\beta H/\sqrt{2}}$$

$$= \sum_N e^{-\beta N/\sqrt{2}R} \int_{U(N)} [dA d\phi^I d\psi]\; e^{-S[A,\phi^I,\psi]}$$

where the 2-D space is a torus with metric

$$ds^2 = |d\sigma_1 + \tau d\sigma_2|^2 \qquad \tau = \frac{\sqrt{2}\beta R}{4\pi\alpha'} i$$

$$\phi^I(\sigma_1 + 1, \sigma_2) = \phi^I(\sigma_1, \sigma_2)$$

$$\phi^I(\sigma_1, \sigma_2 + 1) = \phi^I(\sigma_1, \sigma_2)$$

$$\psi(\sigma_1 + 1, \sigma_2) = \psi(\sigma_1, \sigma_2)$$

$$\psi(\sigma_1, \sigma_2 + 1) = -\psi(\sigma_1, \sigma_2)$$

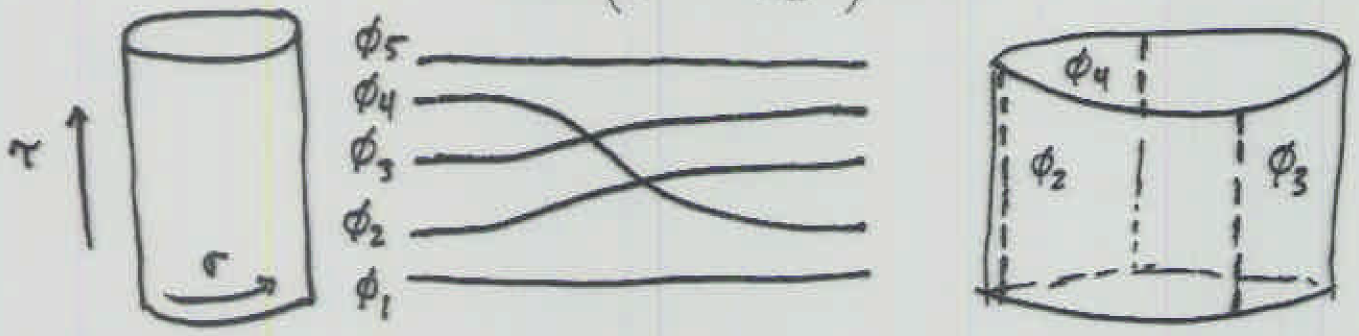
Weak coupling limit of matrix string = strong coupling limit of 2D SYM theory.

Weak string coupling=strong YM coupling limit.

$$\phi^I = U\left(\text{diag}(\phi^I_1, \ldots, \phi^I_N)\right) U^\dagger$$

$$\det\left(\phi^I - \phi^I_a \mathcal{I}\right) = 0$$

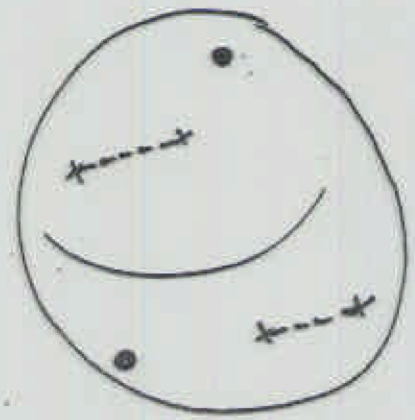
$\phi^I_a \in (R^8)^N / S_N$



Dijkgraaf-Verlinde-Verlinde: Long strings are multiple covers of the cylinder.

Interactions are mediated by instantons which are branched covers of the cylinder.

High energy scattering of strings - sum over Riemann surfaces which are branched covers of the punctured sphere.



In that limit,

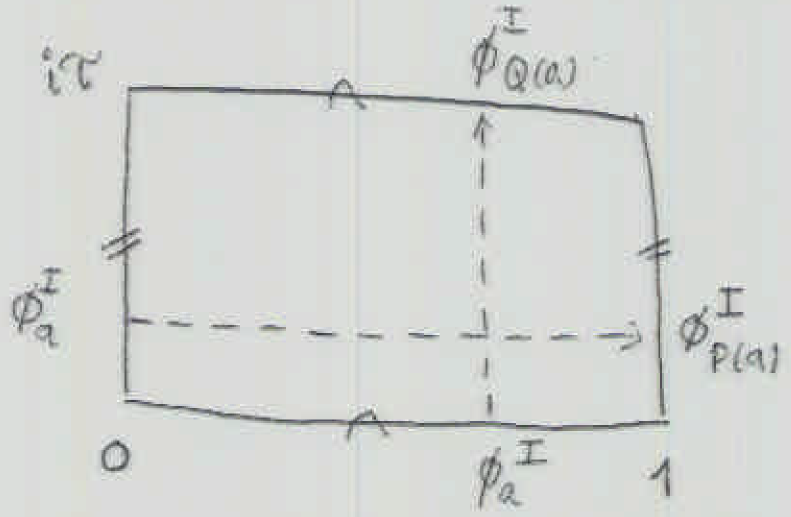
$\phi^I = U\phi_d^I U^\dagger \quad \psi = U\psi_d U^\dagger \quad A_\mu = U\left(a_\mu + i\partial_\mu\right)U^\dagger$

Spectrum is periodic, but eigenvalues can permute,

$$\phi_a^I(\sigma_1 + 1, \sigma_2) = \phi_{P(a)}^I(\sigma_1, \sigma_2)$$

$$\phi_a^I(\sigma_1, \sigma_2 + 1) = \phi_{Q(a)}^I(\sigma_1, \sigma_2)$$

$$PQ = QP$$



Permutations: An element of S_N can be decomposed into cycles. e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 1 & 2 & 5 & 3 & 8 & 9 & 7 \end{pmatrix} =$$

$$= (163)(24)(5)(789)$$

Consider an element of S_N Suppose the number of cycles of length k is r_k: $N = \sum k r_k$.

$(n, k, \alpha) = n$'th element of α'th cycle of length k.

$$P(n, k, \alpha) = (n + 1, k, \alpha)$$

Assume Q comutes with P

$P(Q(n, k, \alpha)) = Q(P(n, k, \alpha)) = (Q(n + 1, k, \alpha))$

$(Q(1, k, \alpha), Q(2, k, \alpha), ..., Q(n, k, \alpha))$ is a cycle of length k. $=((s, k, \pi_k(\alpha)), (s + 1, k, \pi_k(\alpha)), ..., (s + n,$ where $\pi_k(\alpha)$ is a permutation of the cycles of length k. $[Q] = \prod_k r_k! k^{r_k}$

$$\phi_a^I(\sigma_1 + 1, \sigma_2) = \phi_{P(a)}^I(\sigma_1, \sigma_2)$$

$$\phi_a^I(\sigma_1, \sigma_2 + 1) = \phi_{Q(a)}^I(\sigma_1, \sigma_2) \quad (-1)^F$$

relabeled:

$$\phi_{n,k,\alpha}^I(\sigma_1 + 1, \sigma_2) = \phi_{n+1,k,\alpha}^I(\sigma_1, \sigma_2)$$

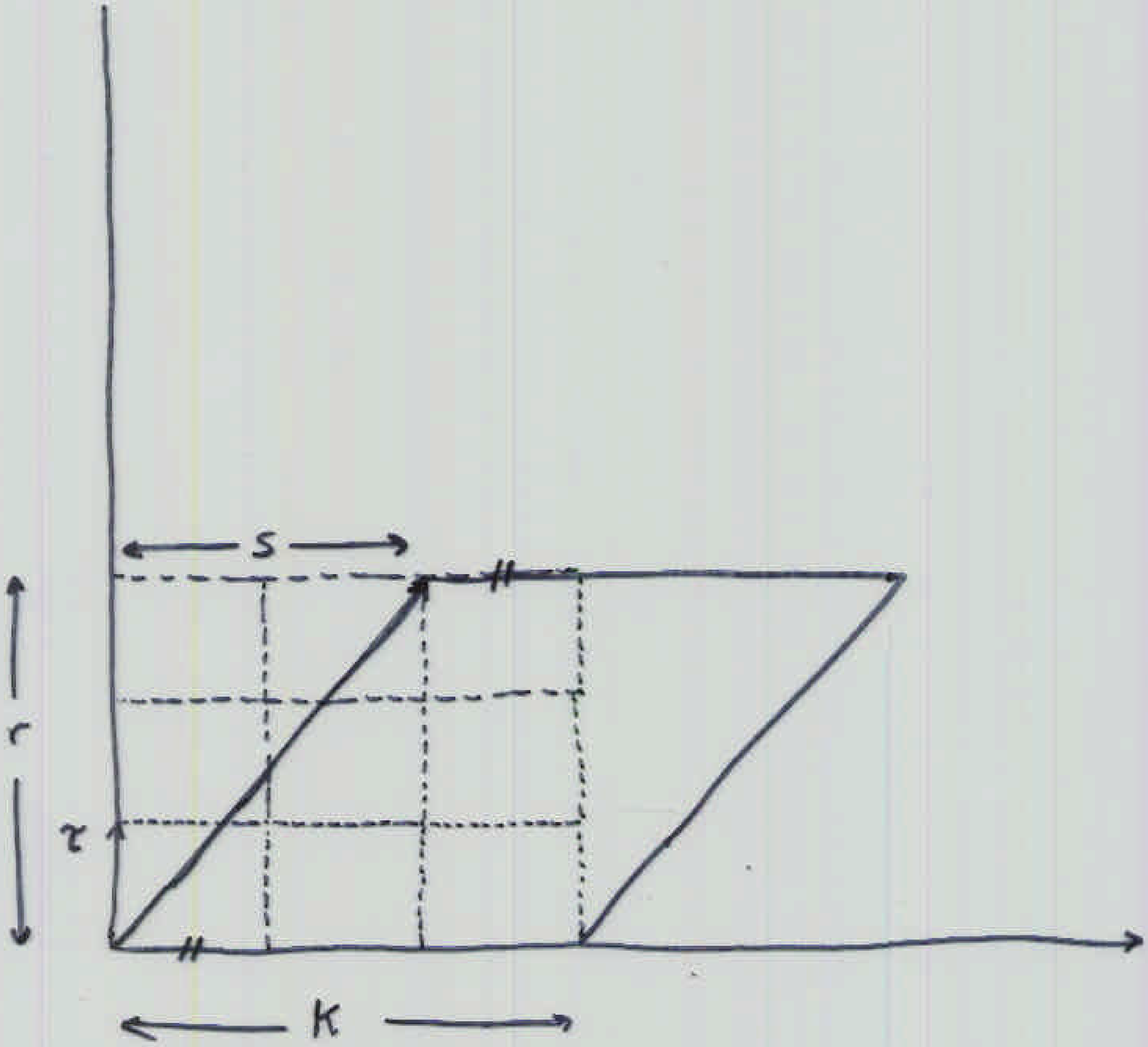
$$\phi_{n,k,\alpha}^I(\sigma_1, \sigma_2 + 1) = \phi_{n,k,\pi_k(\alpha)}^I(\sigma_1 + s(k, \alpha), \sigma_2) \quad (-1)^F$$

Fuse into a single field with

$$\phi_\Omega^I(\sigma_1 + k, \sigma_2) = \phi_\Omega^I(\sigma_1, \sigma_2)$$

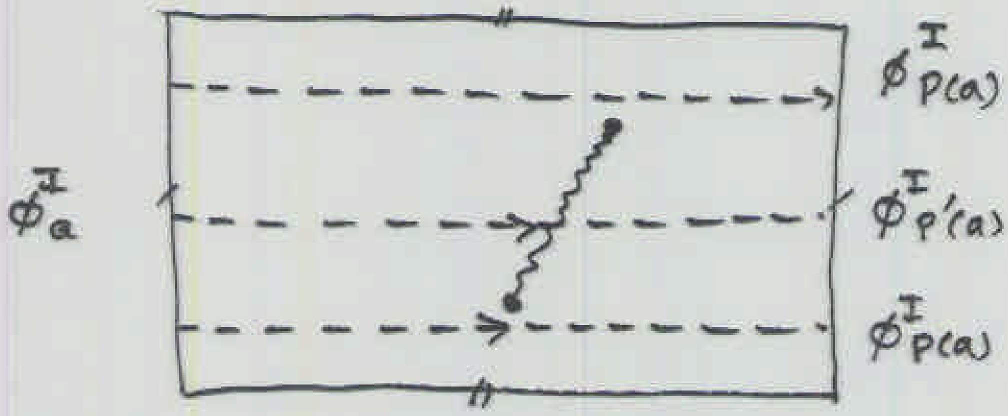
$$\phi_\Omega^I(\sigma_1, \sigma_2 + r) = \phi_\Omega^I(\sigma_1 + s, \sigma_2) \quad (-1)^{Fr}$$

$s = \sum_\alpha s(k, \alpha)$. Torus with $\Omega = \frac{s + \tau r}{k}$



$$\Omega = \frac{s}{k} + \tau \frac{r}{k}$$

This is not the whole story: it obtains only the genus 1 Riemann surfaces. It is possible to get higher genus surfaces as branched covers of the torus:



$$\beta V F = \sum_{\text{surfaces}} e^{-\beta A/\sqrt{2}R}.$$

$$\cdot \int [dad\phi d\psi] e^{-\frac{1}{4\pi\alpha'} \int \left(\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2}\partial\phi\partial\phi + \psi\partial\psi\right)}$$

surfaces are all branched covers of the torus.
$g_s^{-1} = g_{\text{YM}}\sqrt{\alpha'}$