On the Consistency of Non-Critical Non-Orientable Open-Closed String Field Theories

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1 Introduction

- Nonperturbative definition of string theories
  - The large N reduction of the super Yang-Mills theories deduces simple systems of matrix variables as the constructive definition of Type IIA, IIB and Type I superstrings.

    Banks-Fischler-Shenker-Susskind P.R. D55 (1997) 5112
    Ishibashi-Kawai-Kitazawa-Tsuchiya N.P. B498 (1997) 469
    Itoyama-Tsuchiya P.T.P. 101 (1999) 137
- Type I superstrig
  - Heterotic/Type I duality
  - a theory of non-orientable strings
  - Gauge group is introduced by Chan-Paton method.
  - The finiteness of amplitudes and anomaly cancellation specify the gauge group $SO(32)$.
  - The non-orientable bosonic string is also consistent only if the gauge group is $SO(2^{13} = 8192)$.
- Construction of string field theories from matrix models
  - The contimuum limit of the string field theory can be taken at the double scaling limit of the matrix models.

    Ishibashi-Kawai P.L. B314 (1993) 190, B322 (1994) 67
  - Application of Stochastic Quantization Method

    Jevicki-Rodrigues N.P. B421 (1994) 278
    Nakazawa M.P.L. A10 (1995) 2175
- Conformal matter on the orientable 2D surfaces with boundaries
  - loop gas model on orientable 2D surfaces

    Kazakov-Kostov N.P. B386 (1992) 520
    Kostov P.L. B349 (1995) 284
- Construction of non-critical non-orientable open-closed string field theories

- Introduction of Chan-Paton factor for the nonorientable $c = 0$ case

Nakazawa-Ennyu P.L.B417 (1998) 247

- non-orientable 2D random surfaces with "coloured "boundaries are described by the real symmetric matrix-vector model

$$S_0 = \mathrm{Tr}(\frac{1}{2}M^2 - \frac{1}{3}\frac{g}{\sqrt{N}}M^3) + \sum_a V^a(1 - \frac{g_B^a}{\sqrt{N}}M)V^a$$

- The partition function

$$\begin{aligned} Z &= \int dM \Pi_a dV^a e^{-S_0} \\ &= \sum_{DT} \Pi_a^R (g_B^a)^{L_B^a} g^A N^\chi \end{aligned}$$

The sum is taken over all dynamically triangulated 2D non-orientable surfaces with boundaries.
χ is the Eular number
χ = 2 - 2 #(handles) - #(boundaries) - #(crosscaps).

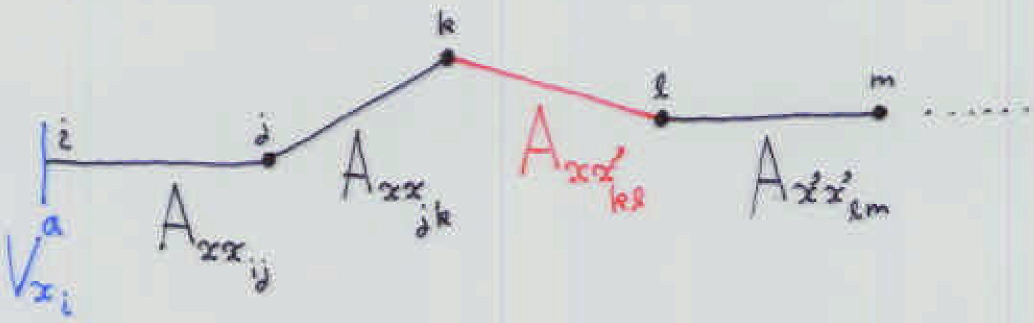
- This talk
  - a loop gas model on non-orientable 2D surfaces
  - Derivation of the string field theory based on SQM
  - Algeblaic structure of string field theory hamiltonian
  - Schwinger-Dyson equation and the consistency of the non-orientable string field theory

2 Real Symmetric Matrix-Vector Model

matrix-vector model ⇒ non-orientable open-closed strings

$A_{xx'\,ij}$: $N \times N$ matrix
$V^a_{x\,i}$: N-vector

$x,\ x' \in \mathbf{Z}$ ← domain in discretized 1-D target space
$a = 1 \sim r$ ← Chan-Paton factor
$(i,\ j = 1 \sim N)$



$$V^a_{x_i} A_{xx_{ij}} A_{xx_{jk}} A_{xx'_{kl}} A_{x'x'_{lm}} \cdots\cdots$$

closed string

open string

Kostov's loop-gas model $\Rightarrow$ conformal matters
on the non-orientable surfaces

$V^a_{x\,i}$

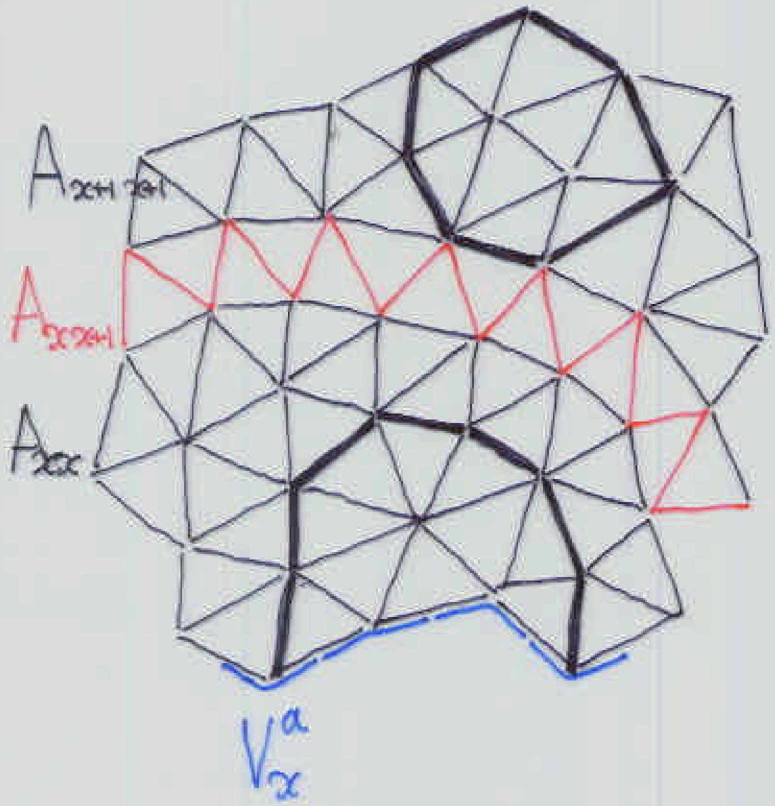
V^a_x : real

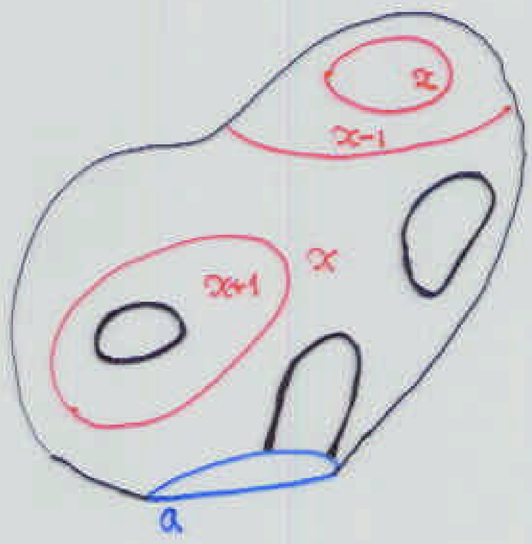
$A_{xx'\,ij}$

$$
\begin{array}{lll}
A_{xx} & : \text{real symmetric} & (x' = x) \\
A_{xx+1} = A^t_{x+1x} & & (x' = x \pm 1) \\
A_{xx'} = 0 & & (x' \neq 0,\ x' \neq x \pm 1)
\end{array}
$$

Action

$$
\begin{aligned}
S \;=\; & \frac{1}{2}\mathrm{tr}\Big(\frac{1}{2}\sum_{x,x'} A_{xx'}A_{x'x} - \frac{1}{3}\frac{g}{\sqrt{N}}\sum_{x,x',x''} A_{xx'}A_{x'x''}A_{x''x}\Big) \\
& + \frac{1}{2}\sum_{x,x'}\sum_{a=1}^{r} V^a_x\Big(\delta_{xx'} - \frac{g^a_B}{\sqrt{N}}A_{xx'}\Big)V^a_{x'}
\end{aligned}
$$





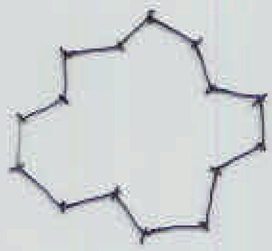
Effective action $\Leftarrow$ Integrating out of $A_{xx\pm 1}$

$S_{\rm eff}(M_x,\ V_x)$

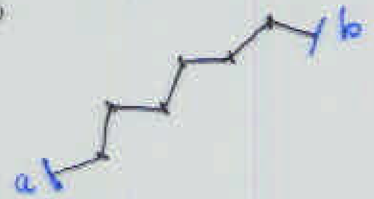
$$M_{x\ ij} = A_{xx\ ij} - \tfrac{\sqrt{N}}{2g}\delta_{ij} \quad : \ \text{"real symmetric"}$$

- String fields : "non-orientable"
  (length L, localized at the site x)

closed string field : $\phi_x(L) \equiv \frac{1}{N}\mathrm{tr}(\mathrm{e}^{L\frac{M_x}{\sqrt{N}}})$



open string field : $\psi_x^{ab}(L) \equiv \frac{1}{N}V^a \mathrm{e}^{L\frac{M_x}{\sqrt{N}}} V^b$



$S_{\rm eff} =$

$$\frac{1}{2}\sum_{x,x'}\int_0^\infty dL\left\{\frac{1}{2L}N^2 C_{xx'}^{(p_0)}\phi_x(L)\phi_{x'}(L) + N\sum_{a,b}\frac{g_B^a g_B^b}{g}C_{xx'}^{(p_0/2)}\psi_x^{ab}(L)\psi_{x'}^{ba}(L)\right\}$$

$$-\frac{N}{2}\mathrm{Tr}\sum_x\left(\frac{1}{2g}+\frac{M_x}{\sqrt{N}}\right)^2 + \frac{Ng}{3}\mathrm{Tr}\sum_x\left(\frac{1}{2g}+\frac{M_x}{\sqrt{N}}\right)^3$$

$$+\sum_x\sum_a\{V_x^a\left(\frac{g_B^a}{2g} - 1 + g_B^a\frac{M_x}{\sqrt{N}}\right)V_x^a\}$$

$$\begin{aligned} C_{xx'}^{(p_0)} &= \delta_{x'x+1} + \delta_{x'x-1} \\ C_{xx'}^{(p_0/2)} &= \delta_{x'x+1} + \delta_{x'x-1} \end{aligned}$$

Adjacency matrices

$$C^{(p_0)}_{xx'} = \cos(\pi p_0)(\delta_{x'x+1} + \delta_{x'x-1})$$
$$C^{(p_0/2)}_{xx'} = \cos(\frac{\pi}{2}p_0)(\delta_{x'x+1} + \delta_{x'x-1})$$

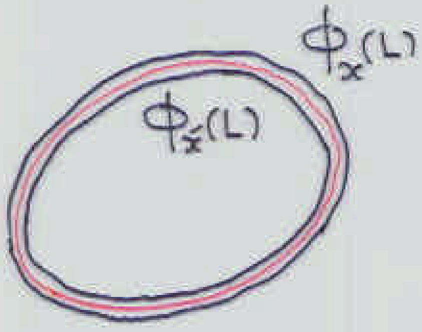
background momentum $p_0 = \frac{1}{m}$

central charge $c = 1 - \frac{6}{m(m+1)}$

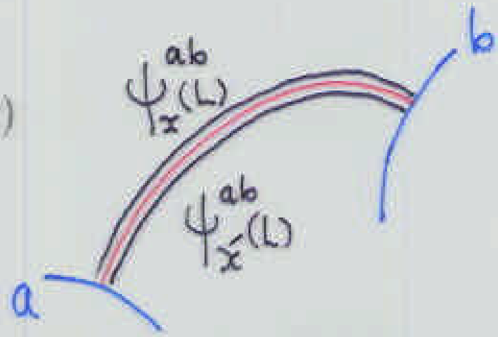
$\Rightarrow \; 0 < c \leq 1$

Interactions

$$S_{\text{eff}} = \int dL \{ \frac{1}{L} C^{(p_0)}_{xx'}$$



$$+ \sum_{a,b} C^{(p_0/2)}_{xx'} \qquad \}$$



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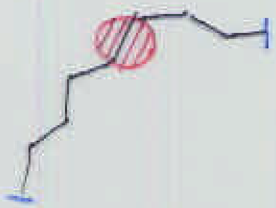
3 Time Evolution of Strings

Langevin equations : one-step deformation of matrices and vectors

$$M_{x\ ij}(\tau+\Delta\tau) \equiv M_{x\ ij}(\tau)+\Delta M_{x\ ij}(\tau)$$
$$V^a_{x\ i}(\tau+\Delta\tau) \equiv V^a_{x\ i}(\tau)+\Delta V^a_{x\ i}(\tau)$$

τ : stochastic time

$$\Delta M_{x\ ij}(\tau) = -\frac{\partial S_{\rm eff}}{\partial M_{xji}}\Delta\tau+\Delta\xi_{x\ ij}(\tau)$$



$$\Delta V^a_{x\ i}(\tau) = -\lambda^a_x\frac{\partial S_{\rm eff}}{\partial V^a_{xi}}\Delta\tau+\Delta\eta^a_{x\ i}(\tau)$$

λ^a_x : scale parameter of the stochastic time evolution on the boundary

Correlation of the white noises $\Delta\xi_{x\ ij}$, $\Delta\eta_{xi}$

$$< \Delta\xi_{x\ ij}(\tau)\Delta\xi_{x'\ kl}(\tau) >_\xi = \Delta\tau\delta_{xx'}(\delta_{il}\delta_{jk}+\delta_{ik}\delta il)$$
$$< \Delta\eta^a_{xi}(\tau)\Delta\eta^b_{x'j}(\cdot\cdot) >_\eta = 2\lambda^a\Delta\tau\delta_{xx'}\delta^{ab}\delta_{ij}$$

Langevin equation for string fields $\rightarrow$ string interactions

- closed string : $\phi_x(L) \equiv \frac{1}{N}\mathrm{tr}(\mathrm{e}^{L\frac{M_x}{\sqrt{N}}})$ 

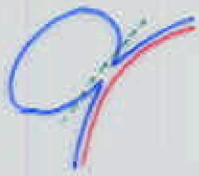
$$\Delta\phi_x(L) = \Delta\tau L\{\frac{1}{2}\int_0^L dL'\phi_x(L')\phi_x(L-L')$$

$$+\frac{1}{2N}L\phi_x(L)$$

$$+\frac{1}{2}\sum_{x'}C^{(p_0)}_{xx'}\int_0^\infty dL'\phi_x(L+L')\phi_{x'}(L')$$

$$+\frac{1}{Ng}\sum_{ab}\sum_{x'}C^{(p_0/2)}_{xx'}\int_0^\infty dL'L'\psi^{ab}_x(L+L')\psi^{ab}_{x'}(L')$$

$$+\frac{1}{N}\sum_a\psi^{aa}(L)$$



$$+(g\frac{\partial^2}{\partial L^2}-\frac{1}{4g})\phi_x(L)\}$$

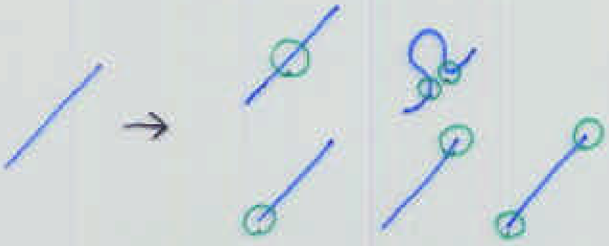
$$+\Delta\zeta_x(L)$$

$$\Delta\zeta_x(L) \equiv \frac{1}{N}L\mathrm{tr}(\mathrm{e}^{L\frac{M_x}{\sqrt{N}}}\frac{\Delta\xi_x}{\sqrt{N}})$$

Noise correlation

$$< \Delta\zeta_x(L)\Delta\zeta_{x'}(L') >= \Delta\tau\delta_{xx'}\frac{2LL'}{N^2}\phi_x(L+L')$$

- open string : $\psi_x^{ab}(L) \equiv \frac{1}{N} V^a e^{L\frac{M_x}{\sqrt{N}}} V^b$

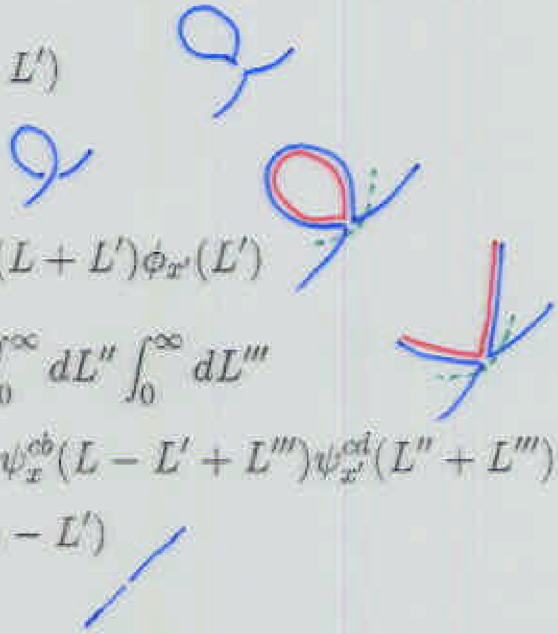


$$\Delta\psi_x^{ab}(L) = 2\lambda^a \Delta\tau \{ \frac{1}{g} \sum_c \sum_{x'} C_{xx'}^{(p_0/2)} \int_0^\infty dL' \psi_x^{bc}(L+L') \psi_{x'}^{ac}(L') + (\frac{\partial}{\partial L} + \frac{1}{2g} - \frac{1}{g_B^a}) \psi_x^{ab}(L) \}$$

$$+ \ (a \leftrightarrow b)$$

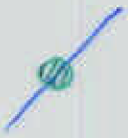
$$+ \ 2\lambda^a \delta^{ab} \delta\tau \phi_x(L)$$

$$\begin{aligned} + \ \Delta\tau \{ & \int_0^L dL' L' \psi_x^{ab}(L') \phi_x(L-L') \\ & + \frac{1}{N}\frac{1}{2} L^2 \psi_x^{ab}(L) \\ & + \frac{1}{2} L \sum_{x'} C_{xx'}^{(p_0)} \int_0^\infty dL' \psi_x^{ab}(L+L') \phi_{x'}(L') \\ & + \frac{1}{g} \sum_{cd} \sum_{x'} C_{xx'}^{(p_0/2)} \int_0^L dL' \int_0^\infty dL'' \int_0^\infty dL''' \\ & \qquad \psi_x^{ad}(L'+L'') \psi_x^{cb}(L-L'+L''') \psi_{x'}^{cd}(L''+L''') \\ & + \sum_c \int_0^L dL' \psi_x^{ac}(L') \psi_x^{cb}(L-L') \\ & + L(\frac{1}{4g} - \frac{\partial}{\partial L}) \psi_x^{ab}(L) \} \end{aligned}$$

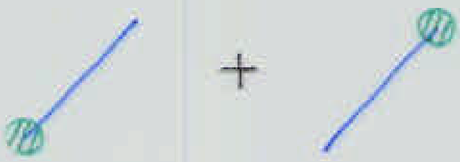


$$+ \ \Delta\zeta_x^{ab}(L) + \Delta\eta_x^{ab}(L)$$

$$\Delta\zeta_x^{ab}(L) = \frac{1}{N} \int_0^L dL' V_x^a e^{L'\frac{M_x}{\sqrt{N}}} \frac{\Delta\xi_x}{\sqrt{N}} e^{(L-L')\frac{M_x}{\sqrt{N}}} V_x^b$$

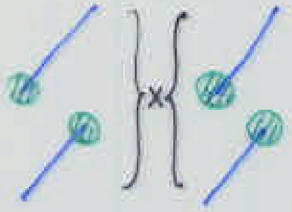


$$\Delta\eta_x^{ab}(L) = \frac{1}{N} (\Delta\eta_x^a e^{L\frac{M_x}{\sqrt{N}}} V_x^b + V_x^a e^{L\frac{M_x}{\sqrt{N}}} \Delta\eta_x^b)$$

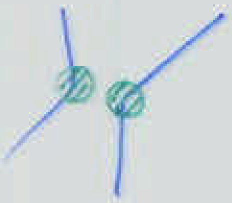


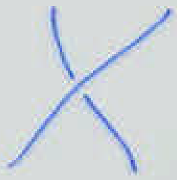
Noise correlations

$$
\begin{aligned}
< \Delta\eta_x^{ab}(L)\Delta_x^{cd}(L') > \;=\;& \lambda^a\delta^{ac}\Delta\tau\frac{2}{N}\psi_x^{bd}(L+L') \\
&+ \lambda^b\delta^{bd}\Delta\tau\frac{2}{N}\psi_x^{ac}(L+L') \\
&+ \lambda^a\delta^{ad}\Delta\tau\frac{2}{N}\psi_x^{bc}(L+L') \\
&+ \lambda^b\delta^{bc}\Delta\tau\frac{2}{N}\psi_x^{ad}(L+L')
\end{aligned}
$$



$$
\begin{aligned}
< \Delta\zeta_x^{ab}(L)\Delta\zeta_x^{cd}(L') > \;=\;& \Delta\tau\frac{1}{N}\int_0^L ds\int_0^{L'} ds' \\
& \{\psi_x^{ad}(s+s')\psi_x^{cd}(L+L'-s-s') \\
& +\psi_x^{ac}(s+s')\psi_x^{bd}(L+L'-s-s')\}
\end{aligned}
$$





$$
< \Delta\zeta_x(L)\Delta\zeta_x^{ab}(L') > \;=\; \Delta\tau\frac{2LL'}{N^2}\psi_x^{ab}(L+L')
$$

4 Fokker-Planck Hamiltonian

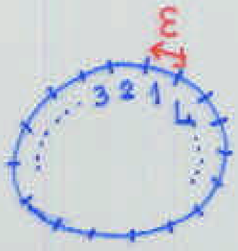
F-P Hamiltonian Operator $\hat{H}_{\mathrm{FP}}$

$$< \phi(0), \psi(0)| e^{-\tau \hat{H}_{\mathrm{FP}}} O(\hat{\phi}, \hat{\psi})|0 > \equiv < O(\phi_{\xi\eta}(\tau), \psi_{\xi\eta}(\tau)) >_{\xi\eta}$$

$O(\phi, \psi)$	:	observable
$\phi_{\xi\eta}(\tau)$, $\psi_{\xi\eta}(\tau)$	:	solutions of the Langevin equations
$< \ >_{\xi\eta}$	:	noise correlation
$\phi_x(0)$, $\psi_x(0)$	:	initial values

$$
\begin{aligned}
&-\Delta\tau < \hat{H}_{\mathrm{FP}}(\tau) O(\phi, \psi) > \\
&\quad = \ < \Delta O > |_{O(\Delta\tau)} \\
&\quad = \ < \ \int_0^\infty dL \Delta\phi_x(L) \frac{\partial O}{\partial \phi_x(L)} \\
&\qquad + \sum_{ab} \int_0^\infty dL \Delta\psi_x^{ab}(L) \frac{\partial O}{\partial \psi_x^{ab}(L)} \\
&\qquad + \frac{1}{2} \int_0^\infty dL \int_0^\infty dL' \Delta\zeta_x(L) \Delta\zeta_x(L') \frac{\partial^2 O}{\partial \phi_x(L) \partial \phi_x(L')} \\
&\qquad + \frac{1}{2} \sum_{abcd} \int_0^\infty dL \int_0^\infty dL' \Delta\zeta_x^{ab}(L) \Delta\zeta_x^{cd}(L') \frac{\partial^2 O}{\partial \psi_x^{ab}(L) \partial \psi_x^{cd}(L')} \\
&\qquad + \sum_{ab} \int_0^\infty dL \int_0^\infty dL' \Delta\zeta_x(L) \Delta\zeta_x^{ab}(L') \frac{\partial^2 O}{\partial \phi_x(L) \partial \psi_x^{ab}(L')} \quad > |_{O(\Delta\tau)}
\end{aligned}
$$

5 Continuum Limit



"ε" : the minimal unit of length

$\ell \equiv L\varepsilon$	:	the physical length of strings
$d\tau \equiv \epsilon^{-2-D}\Delta\tau$	:	infinitesimal stochastic time
$\lambda^a \equiv \varepsilon^{-1/2-D/2}\lambda_x^a$	:	the scaling of stochastic time on boundaries

Renormalized field operators

$$\Phi_x(\ell) \equiv \varepsilon^{-D}\phi_x(L) \qquad \Psi_x^{ab}(\ell) \equiv \varepsilon^{-1/2-D/2}\psi_x^{ab}(L)$$
$$\Pi_x(\ell) \equiv \varepsilon^{-1+D}\frac{\partial}{\partial\phi_x(L)} \qquad \Pi_x^{ab}(\ell) \equiv \varepsilon^{-1/2+D/2}\frac{\partial}{\partial\psi_x^{ab}(L)}$$

Double scaling limit : $\varepsilon \to 0,\ N \to \infty$

$G_{st} \equiv N^{-2}\varepsilon^{-2D}$ (finite)	:	string coupling constant
$g^* - g \sim a^2\Lambda$		(Λ : cosmological constant)
$g_B^{a*} - g_B^a \sim a^{p_0+\frac{1}{2}}\mu$		(μ : mass at the string end point)

$$D = 2 + p_0 = 2 + \tfrac{1}{m}$$

Commutation relations

$$[\Pi_x(\ell), \Phi_{x'}(\ell')] = \delta_{xx'}\delta(\ell - \ell')$$
$$[\Pi_x^{ab}(\ell), \Psi_{x'}^{cd}(\ell')] = \frac{1}{2}(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})\delta_{xx'}\delta(\ell - \ell')$$

Continuum limit of the F-P hamiltonian

$$\begin{aligned}\mathcal{H}_{\mathrm{FP}} = \sum_x \int_0^\infty d\ell \Big\{ & -G_{\mathrm{st}}\mathcal{L}_x(\ell)\,\ell\,\Pi_x(\ell) \qquad (1)\\ & -\sqrt{G_{\mathrm{st}}}\sum_{a,b}^{r}\Big(\lambda^a\mathcal{J}_x^{ab}(\ell)+\lambda^b\mathcal{J}_x^{ba}(\ell)\Big)\Pi_x^{ab}(\ell) \qquad (2)\\ & +\sum_{a,b}^{r}\mathcal{K}_x^{ab}(\ell)\,\ell\,\Pi_x^{ab}(\ell) \qquad (3)\\ & +\sqrt{G_{\mathrm{st}}}\sum_{a,b,c}^{r}\int_0^\ell d\ell'\ell'\mathcal{J}_x^{cb}(\ell')\Psi_x^{ac}(\ell-\ell')\Pi_x^{ab}(\ell)\Big\}\end{aligned}$$

Three generators

(1) closed string

$\rightarrow \quad \mathcal{L}_x(\ell)$

(2) open string $\qquad \mathcal{J}_x^{ab}(\ell) \qquad \mathcal{J}_x^{ba}(\ell)$

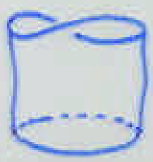
(3) open string $\qquad \mathcal{K}_x^{ab}(\ell)$

(1)

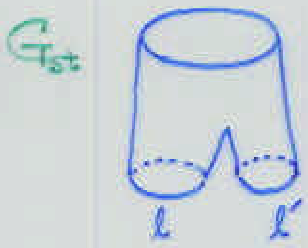
$\rightarrow \quad \mathcal{L}_x(\ell)$

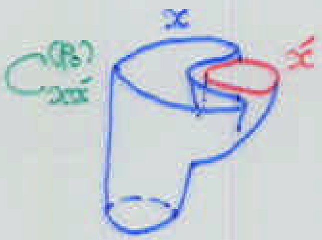
$$-G_{\rm st}\mathcal{L}_x(\ell) = \frac{1}{2}\int_0^{\ell} d\ell'\Phi_x(\ell')\Phi_x(\ell-\ell')$$

$$+\frac{1}{2}\sqrt{G_{\rm st}}\;\ell\;\Phi_x(\ell)$$

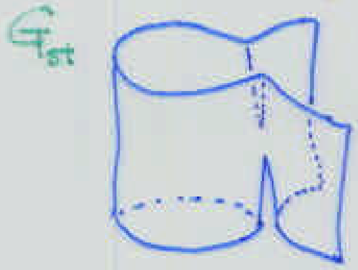


$$+\int_0^{\infty} d\ell'\Phi_x(\ell+\ell')\Big\{G_{\rm st}\ell'\Pi_x(\ell') + \frac{1}{2}\sum_{x'} C^{(p_0)}_{xx'}\Phi_{x'}(\ell')\Big\}$$





$$+\sqrt{G_{\rm st}}\sum_{a,b}^{r}\int_0^{\infty} d\ell'\,\ell'\,\Psi^{ab}_x(\ell+\ell')\Big\{\sqrt{G_{\rm st}}\Pi^{ab}_x(\ell') + \frac{1}{g}\sum_{x'} C^{(p_0/2)}_{xx'}\Psi^{ba}_{x'}(\ell')\Big\}$$



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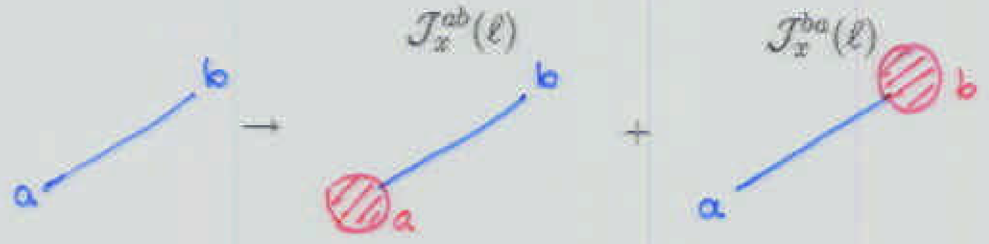
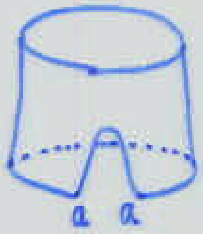
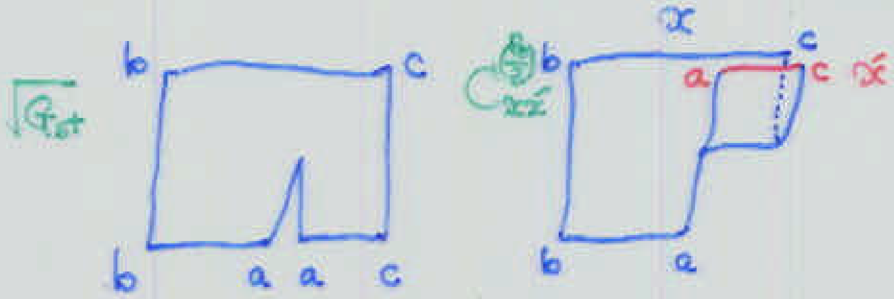


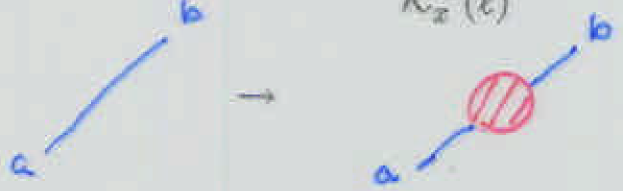
$$-\sqrt{G_{\rm st}}\mathcal{J}_x^{ab}(\ell) \;=\; \frac{1}{2}\Phi_x(\ell)\delta^{ab}$$

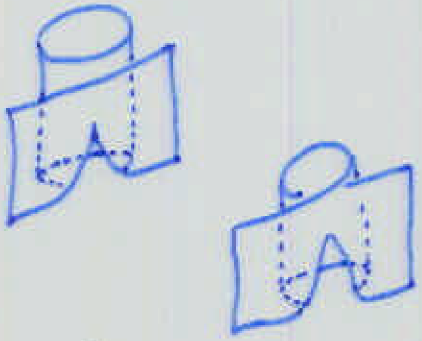


$$+\sum_c^r \int_0^\infty d\ell' \Psi_x^{cb}(\ell+\ell')\Big\{\sqrt{G_{\rm st}}\Pi_x^{ca}(\ell') + \frac{1}{g}\sum_{x'} C_{xx'}^{(p_0/2)}\Psi_{x'}^{ca}(\ell')\Big\}$$

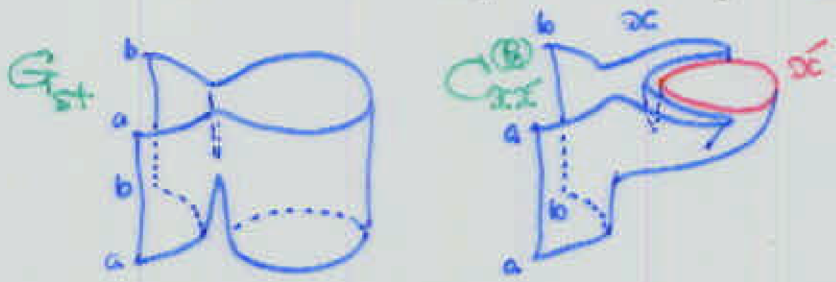


$\mathcal{K}_x^{ab}(\ell)$

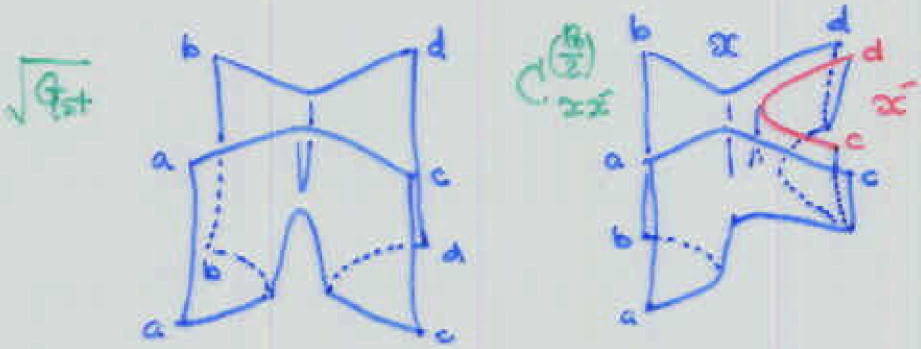




$$\mathcal{K}_x^{ab}(\ell) = \int_0^\ell d\ell' \Psi_x^{ab}(\ell')\Phi_x(\ell-\ell')$$
$$+\frac{1}{2}\sqrt{G_{\rm st}}(r+2)\ell\Psi_x^{ab}(\ell)$$
$$+ \int_0^\infty d\ell' \Psi_x^{ab}(\ell+\ell')\Big\{G_{\rm st}\ell'\Pi_x(\ell') + \frac{1}{2}\sum_{x'} C_{xx'}^{(p_0)}\Phi_{x'}(\ell')\Big\}$$



$$+\sum_{c,d}^{r}\int_0^\infty d\ell' \int_0^{\ell+\ell'} d\ell'' \Psi_x^{ac}(\ell'')\Psi_x^{db}(\ell+\ell'-\ell'')\Big\{\sqrt{G_{\rm st}}\Pi_x^{cd}(\ell') + \frac{1}{g}\sum_{x'} C_{xx'}^{(p_0/2)}\Psi_{x'}^{cd}(\ell')\Big\}$$



6 Algebraic Structure of the Generators

$$[\mathcal{L}_x(\ell), \mathcal{L}_{x'}(\ell')] = (\ell - \ell')\delta_{xx'}\mathcal{L}_x(\ell + \ell')$$

$$[\mathcal{L}_x(\ell), \mathcal{J}^{ab}_{x'}(\ell')] = -\ell'\delta_{xx'}\mathcal{J}^{ab}_x(\ell + \ell')$$

$$[\mathcal{J}^{ab}_x(\ell), \mathcal{J}^{cd}_{x'}(\ell')] = \delta^{cb}\delta_{xx'}\mathcal{J}^{ad}_x(\ell + \ell') - \delta^{ad}\delta_{xx'}\mathcal{J}^{cb}_x(\ell + \ell')$$

$$\begin{aligned}[\mathcal{L}_x(\ell), \mathcal{K}^{ab}_{x'}(\ell')] &= (\ell - \ell')\delta_{xx'}\mathcal{K}^{ab}_x(\ell + \ell') \\ &\quad + \tfrac{\sqrt{G_{st}}}{2}\delta_{xx'}\textstyle\sum_c \int_0^\ell du(\ell - u)\{\mathcal{J}^{cb}_x(\ell + \ell' - u)\Psi^{ac}_x(u) \\ &\qquad + \mathcal{J}^{ca}_x(\ell + \ell' - u)\Psi^{bc}_x(u)\}\end{aligned}$$

$$\begin{aligned}[\mathcal{J}^{ab}_x(\ell), \mathcal{K}^{cd}_{x'}(\ell')] &= -\delta^{ad}\delta_{xx'}\mathcal{K}^{bc}_x(\ell + \ell') - \delta^{ac}\delta_{xx'}\mathcal{K}^{bd}_x(\ell + \ell') \\ &\quad - \tfrac{\sqrt{G_{st}}}{2}\delta_{xx'}\textstyle\int_0^\ell du\{\mathcal{J}^{ad}_x(\ell + \ell' - u)\Psi^{cb}_x(u) + \mathcal{J}^{ac}_x(\ell + \ell' - u)\Psi^{db}_x(u) \\ &\qquad + \textstyle\sum_e\{\delta^{ad}\mathcal{J}^{ec}_x(\ell + \ell' - u)\Psi^{be}_x(u) + \delta^{ac}\mathcal{J}^{ed}_x(\ell + \ell' - u)\Psi^{be}_x(u)\}\}\end{aligned}$$

$$\begin{aligned}[\mathcal{K}^{ab}_x(\ell), \mathcal{K}^{cd}_{x'}(\ell')] &= \tfrac{G_{st}}{4}\delta_{xx'}\textstyle\sum_e \int_0^{\ell'} du \int_0^{\ell'-u} dv \\ &\quad \{\,\mathcal{J}^{ac}_x(\ell{+}\ell'{-}u{-}v)\{\Psi^{cb}_x(u)\Psi^{ed}_x(v) + \Psi^{ce}_x(u)\Psi^{db}_x(v)\} \\ &\quad + \mathcal{J}^{be}_x(\ell{+}\ell'{-}u{-}v)\{\Psi^{ca}_x(u)\Psi^{ed}_x(v) + \Psi^{ce}_x(u)\Psi^{da}_x(v)\} \\ &\quad - \mathcal{J}^{ce}_x(\ell{+}\ell'{-}u{-}v)\{\Psi^{ad}_x(u)\Psi^{eb}_x(v) + \Psi^{ae}_x(u)\Psi^{db}_x(v)\} \\ &\quad - \mathcal{J}^{de}_x(\ell{+}\ell'{-}u{-}v)\{\Psi^{ac}_x(u)\Psi^{eb}_x(v) + \Psi^{ae}_x(u)\Psi^{cb}_x(v)\}\,\}\end{aligned}$$

- the consistency condition for the constraints on the equilibrium expectation value
  ("integrable condition" of stochastic time evolution)

7 Schwinger-Dyson Equation

The Laplace transformed variables

$$\tilde{W}_x(z) \equiv \frac{1}{N} \mathrm{tr} \frac{1}{z - \frac{M_x}{\sqrt{N}}}$$

$$\tilde{\Omega}^{ab}_x(z) \equiv \frac{1}{N} V^a_x \frac{1}{z - \frac{M_x}{\sqrt{N}}} V^b_x$$

$$\frac{1}{2}\tilde{W}_x(z)^2 + \frac{1}{2}\sum_{x'} C^{(P_0)}_{xx'} \int \frac{dz'}{2\pi i} \frac{1}{z-z'} \tilde{W}_x(z') \tilde{W}_{x'}(-z')$$

$$-\frac{1}{2}\frac{1}{N}\partial_z \tilde{W}_x(z) + \frac{1}{Ng}\sum_{x'} C^{(\frac{P_0}{2})}_{xx'} \sum_{ab} \int \frac{dz'}{2\pi i} \frac{1}{(z-z')^2} \tilde{\Omega}^{ab}_x(z') \tilde{\Omega}^{ba}_{x'}(-z')$$

$$+ \text{potential term} = 0$$

$$\delta^{ab}\tilde{W}_x(z) + \frac{2}{g}\sum_{x'} C^{(\frac{P_0}{2})}_{xx'} \sum_{c} \int \frac{dz'}{2\pi i} \frac{1}{z-z'} \tilde{\Omega}^{ac}_x(z') \tilde{\Omega}^{cb}_{x'}(-z')$$

$$+ \text{potential term} = 0$$

- Large N limit ⟶ orientable string case

The solutions of the S-D equations give the correct disc amplitudes.

8 Conclusion

- We have proposed the real symmetric matrix-vector model defined on the discretized 1D target space which describes the conformal matter with the central charge $0 < c \leq 1$ living on the non-orientable 2D random surfaces with boundaries.

- The non-critical non-orientable open-closed string field theory with Chan-Paton facotr is derived from the underlying stochastic process defined by the matrix-vector model.

- F-P hamiltonian is a linear combination of three constraints, $\mathcal{K}_x^{ab}(\ell)$, $\mathcal{L}_x(\ell)$ and $\mathcal{J}_x^{ab}(\ell)$. They satisfy the algebraic relation including the Virasoro and $SO(r)$ current algebras. The closure of the constraints implies the integrability of the time evolution of the underlying stochastic process.

- The large N limit of the S-D equations are consistent to the orientable case. The scaling behaviour belongs to the same universality class as one for the orientable case.

- The conjecture is that, at the central charge $c \to 1$ limit, the non-critical string may be equivalent to 2D string theory. The non-orientable open-closed string theory with $SO(R)$ gauge symmetry by the Chan-Paton method is consistent only if the gauge group is $SO(2)$. To prove the conjecture, we have to investigate more carefully the cancellation of the logarithmic singularities in the disc and the mobius amplitudes.