Perturbative ultraviolet and infrared dynamics of noncommutative quantum field theory

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§1. Topic

A few remarkable properties of quantum dynamics possessed by noncommutative (NC) field theory

with a comment on its implication to string theory

Plan:

§2. Motivation

§3. Perturbative NC field theory

UV-limit of NC field theory $\equiv$ UV and planar limit of the corresponding large-N field theory

§4. IR-aspect of NC field theory

IR-limit $\longleftrightarrow$ UV-limit closely related

§5. Conclusion and discussion

§2. Motivation

Noncommutative field theory appears as one sector of the various matrix models

(Connes, Douglas & Schwarz)

- M(atrix) theory $\Rightarrow$ M-theory (?)
  (Banks, Fischler, Shenker & Susskind)
- IIB matrix model $\Rightarrow$ IIB string (?)
  (Periwal
  Ishibashi, Kawai, Kitazawa & Tsuchiya)
- Matrix string theory $\Rightarrow$ IIA string (?)
  (Motl
  Dijkgraaf, Verlinde & Verlinde)

... intends to provide

the constructive definition of

superstring nonperturbatively ... g_s

However, for instance,

... have no dimensionless parameters

$\Rightarrow$ Perturbative analysis is not available

One idea to overcome such a difficulty:

… Let's consider BPS solution:

$$\begin{cases} [X^\mu, X^\nu] = -i\, C^{\mu\nu}\, 1_N \\ (\mu, \nu = 1, \cdots, 4) \end{cases}$$

$$X^i = 0 \quad (i = 5, \cdots, 10)$$

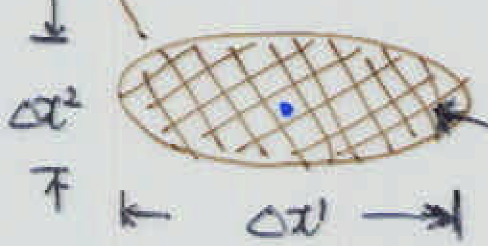
$\left(X^M (M = 1, \cdots, 10) : \text{bosonic } N \times N \text{ matrix variables } (N \to \infty) \right)$

and investigate the theory around this solution, derived from IIB matrix model

(Aoki, Ishibashi, Iso, Kawai, Kitazawa & Tada)

$\boxed{\text{eigenvalues of } X^M = \text{points of universe}}$

$$\boxed{\Delta x^\mu \Delta x^\nu \gtrsim 2\pi |C^{\mu\nu}|}$$



$C^{\mu\nu} = -C^{\nu\mu}$ … (length)^2

area $\gtrsim 2\pi |C^{12}|$

We can express the theory around the solution in terms of field theory

$\Rightarrow$ $\boxed{\text{Noncommutative } \mathcal{N}=4 \text{ supersymmetric Yang-Mills theory in 4-dimension}}$

$$S_{NCYM} = \int d^4x \left[-\frac{1}{4 g_{YM}^2} F_{\mu\nu} * F^{\mu\nu} + \cdots \right]$$

star product

$$(f * g)(x) = f(x) \exp\left[\frac{1}{2i} \overleftarrow{\partial}_\mu C^{\mu\nu} \overrightarrow{\partial}_\nu \right] g(x)$$

$$[X^\mu, X^\nu] = -i C^{\mu\nu} \mathbf{1}_N$$

$C^{\mu\nu}$ characterizes the noncommutativity of product of functions.

(isomorphic)

e.g.)

$$x^\mu * x^\nu - x^\nu * x^\mu = -i C^{\mu\nu}$$

$$\boxed{g_{YM}^2 = \frac{4\pi^2 g_{IIB}^2}{|C^{\mu\nu}|^2}}$$

$$S_{IIB} = -\frac{1}{g_{IIB}^2} \mathrm{Tr}\left[\frac{1}{4}[X^M, X^N][X_M, X_N] + \cdots \right]$$

$$|C^{\mu\nu}| \gg g_{IIB} \Rightarrow g_{YM} \ll 1$$

We can analyze

dynamics of one sector of

IIB matrix model

perturbatively (in g_{YM})

through

NC $\mathcal{N}=4$ Yang-Mills theory

Before that,

simpler systems

$\Rightarrow$ fundamental structures of NC field theory

It may become possible to ask whether

NC $\mathcal{N}=4$ supersymmetric YM theory becomes

quantum theory of gravity

and string theory

§3. Perturbative noncommutative quantum field theory

$\equiv$ summing over Feynman diagrams perturbatively.

(e.g.) real scalar ϕ with action:

$$S_{\phi^4_4} = \int d^4x \left[\frac{1}{2} \partial_\mu\phi \star \partial_\mu\phi + \frac{1}{2} m^2 \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right]$$

It is convenient to work in momentum space:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \tilde{\phi}(p)$$

$\Rightarrow$ Basic product:

$$\underline{e^{ip\cdot x} \star e^{iq\cdot x}}$$

$$= e^{ip\cdot x} e^{\frac{1}{2i} \overleftarrow{\partial}_\mu C^{\mu\nu} \overrightarrow{\partial}_\nu} e^{iq\cdot x}$$

$$= \underline{\boxed{e^{\frac{i}{2} p\wedge q}} e^{i(p+q)\cdot x}}$$

$$p \wedge q \equiv p_\mu C^{\mu\nu} q_\nu$$

$$= -q \wedge p$$

- propagator

momentum conservation $\Rightarrow$ only p

$$\Rightarrow \quad e^{\frac{i}{2} p \wedge p} = 1$$

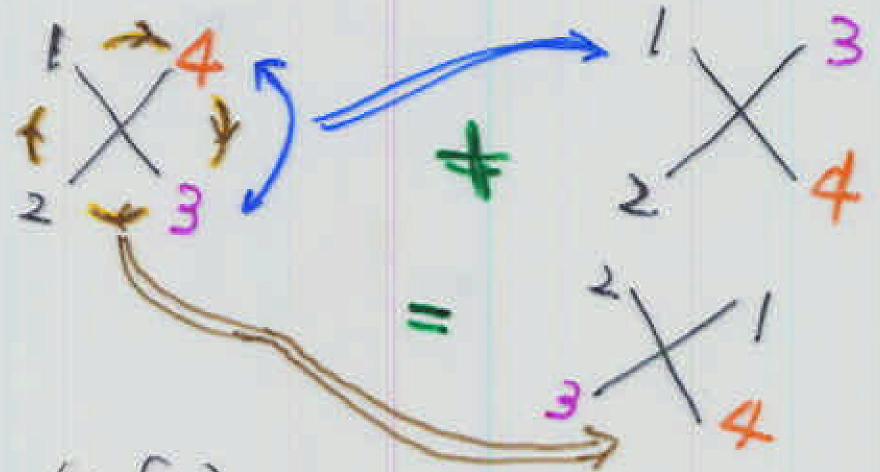
$$\text{——} = \frac{1}{p^2 + m^2}$$

the same as in ordinary field theory

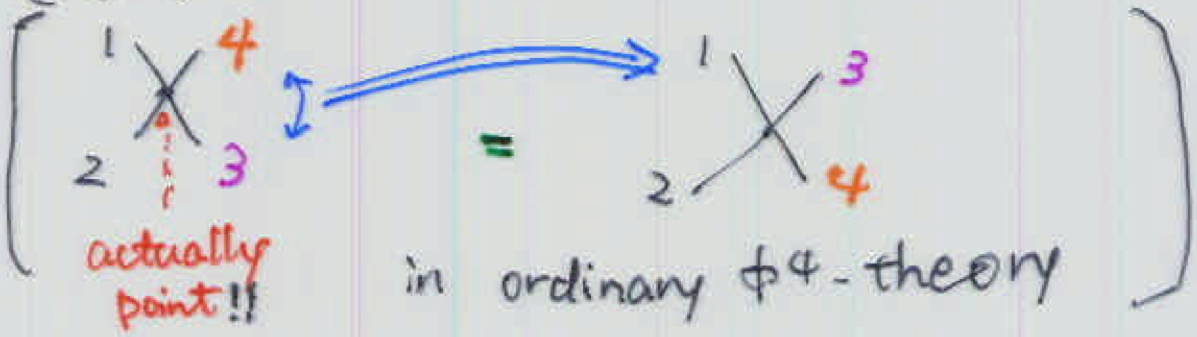
- interaction vertex

$$\times \Leftrightarrow \int \prod_{j=1}^{4} \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_4)$$

$$\times \frac{\lambda}{4} \boxed{e^{\frac{i}{2} \sum_{i<j} p_i \wedge p_j}} \tilde{\Phi}(p_1) \cdots \tilde{\Phi}(p_4)$$

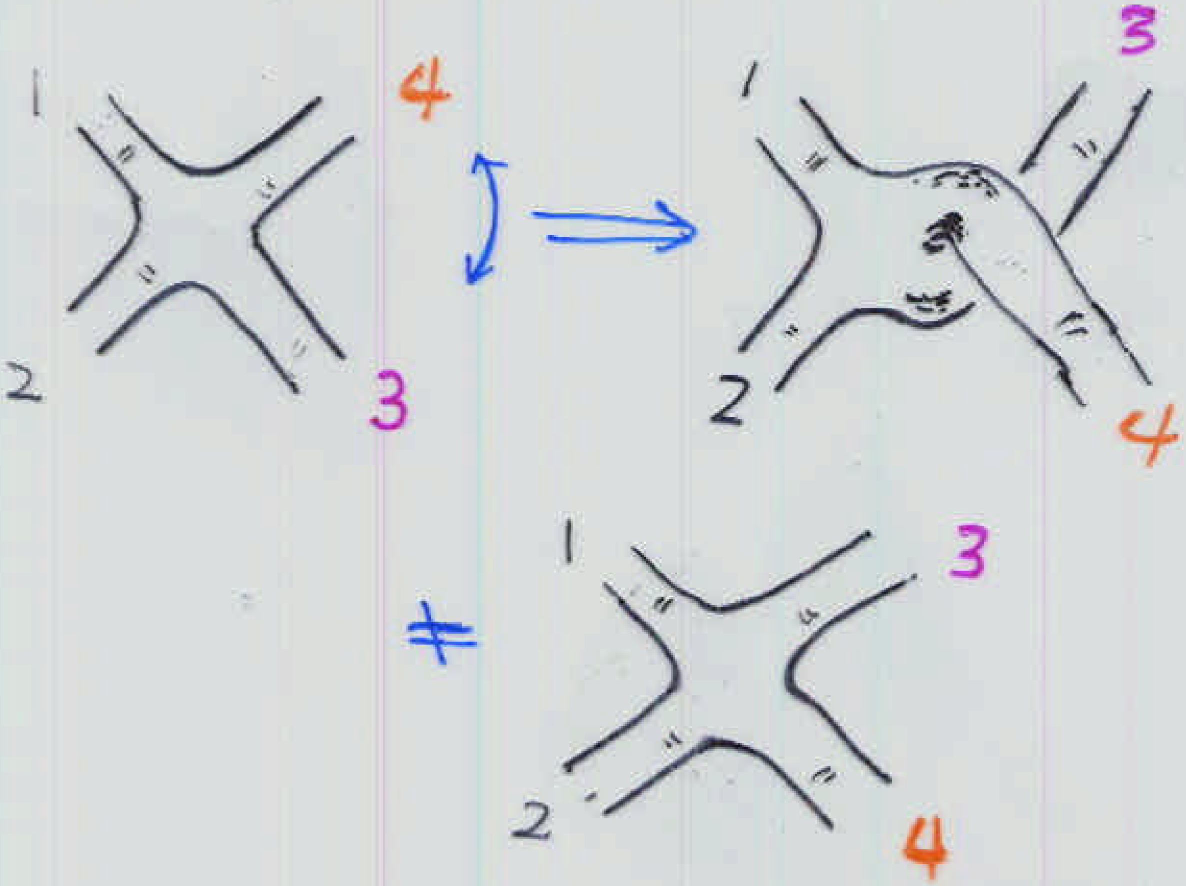


(c.f.)



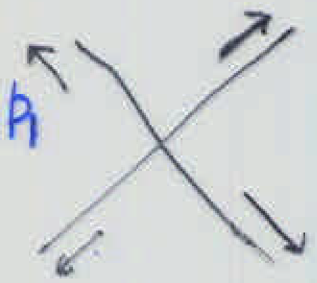
in ordinary ϕ^4-theory

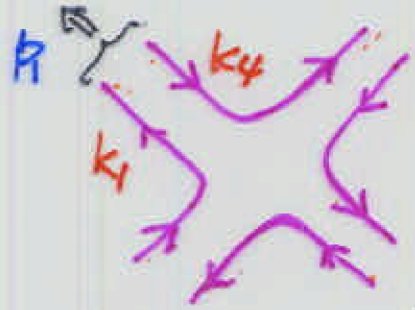
Better picture would be



To pursue the best picture, we write

$$p_j = \underbrace{k_j}_{\text{outgoing momentum}} - \underbrace{k_{j-1}}_{\text{incoming momentum}} \qquad (k_0 \equiv k_4)$$





$$\int \underset{\text{Jacobian factor } (p \to k)}{4^4} \prod_{j=1}^{4} \frac{d^4 k_j}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \cdots + k_4)$$

$$\times \frac{\lambda}{4} \left[e^{\frac{i}{2} k_4 \wedge k_1} \tilde{\Phi}(k_1 - k_4) \right] \times \cdots\cdots \times \underbrace{\left[e^{\frac{i}{2} k_3 \wedge k_4} \tilde{\Phi}(k_4 - k_3) \right]}_{\shortparallel \atop \phi[k_3, k_4]}$$

① $\sum_{i<j} p_i \wedge p_j$

$= \sum_i k_{i-1} \wedge k_i$

$= \boxed{k_4 \wedge k_1} + k_1 \wedge k_2 + k_2 \wedge k_3 + k_3 \wedge k_4$

② $\tilde{\Phi}(p_1) = \boxed{\tilde{\Phi}(k_1 - k_4)}$, ……

$$\boxed{(\phi[k_1, k_2])^* = \phi[k_2, k_1]} \Leftarrow \left(\begin{array}{c} \phi(x) : \text{real} \\ (\tilde{\Phi}(p))^* = \tilde{\Phi}(-p) \end{array} \right)$$

i.e. ϕ is "hermitian" matrix

We are inclined to recall
the ordinary <u>hermitian matrix theory</u>:

$$S[\Phi^4_4]_M = \int d^4x \ \mathrm{tr}\left[\frac{1}{2}\partial_\mu\Phi\,\partial_\mu\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda_4}{4N}\Phi^4\right]$$

$\Phi_i{}^j(x)$: $N \times N$ hermitian matrix-valued field

prepared for the future purpose
to take **large N limit**

In terms of $\tilde{\Phi}(p) = \int d^4x\, e^{-ip\cdot x}\,\Phi(x)$,
the interaction vertex (—) takes the form

$$\boxed{\begin{aligned}&\int 4^4 \prod_{j=1}^{4}\frac{d^4k_j}{(2\pi)^4}(2\pi)^4\delta^4(k_1+\cdots+k_4)\\ &\times\frac{\lambda_4}{4N}\,\tilde{\Phi}_{i_4}{}^{i_1}(k_1-k_4)\cdots\tilde{\Phi}_{i_3}{}^{i_4}(k_4-k_3)\end{aligned}}$$

$\Leftrightarrow$ In NC field theory,

$$\boxed{\begin{aligned}&\int 4^4 \prod_{j=1}^{4}\frac{d^4k_j}{(2\pi)^4}(2\pi)^4\delta^4(k_1+\cdots+k_4)\\ &\times\frac{\lambda}{4}\left[e^{\frac{i}{2}k_4\wedge k_1}\tilde{\phi}(k_1-k_4)\right]\cdots\left[e^{\frac{i}{2}k_3\wedge k_4}\tilde{\phi}(k_4-k_3)\right]\end{aligned}}$$

2x 
\+ 1x 
\+ ...

Feynman diagrams (i.e. their topologies in double-line representation)
in both theories
are completely the same,
including their combinatoric factors

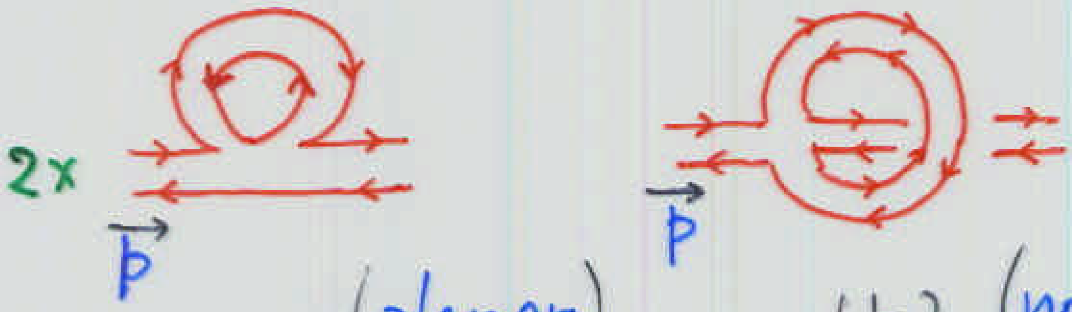
How about the contributions ?

— The phase factor $\left(e^{\frac{i}{2}k\wedge q}\right)$ in NC field theory plays the role of the color indices (carried by $\Phi_i^{\,j}(k-q)$) in the large N field theory.

In fact,

the phase factor distinguishes the planar and the nonplanar diagrams.

(Ex.) one loop contribution to the two point function



(a) (planar) (b) (nonplanar)

(c.f.) large-N field theory side:

$$(a) = a\, \lambda_4 \Lambda_{cut}^2$$

$$(b) = b \cdot \frac{1}{N} \lambda_4 \Lambda_{cut}^2$$

(numerical constants)

(Λ_{cut}: UV-cutoff)

$$\boxed{(a) \gg (b) \quad \text{in the large-}N\text{ limit}}$$

common factor

In NC-field theory,

$$(a) = -2 \times \frac{\lambda}{4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2+m^2} = a\, \lambda \Lambda_{cut}^2$$

$$(b) = -\frac{\lambda}{4} \int \frac{d^4 q}{(2\pi)^4} \boxed{e^{i p \wedge q}} \frac{1}{q^2+m^2}$$

What is the effect of such a nontrivial phase factor?

$$= -\frac{\lambda}{4}\int \frac{d^4q}{(2\pi)^4}\, e^{ip\wedge q} \boxed{\frac{1}{q^2+m^2}}$$

$$= \int_0^\infty d\alpha\, e^{-\alpha(q^2+m^2)}$$

$$= -\frac{\lambda}{4}\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}\, e^{-(\alpha m^2 + \frac{1}{4}\frac{1}{\alpha}\tilde{p}^2)}$$

$$\boxed{\tilde{p}^\mu = C^{\mu\nu} p_\nu}$$

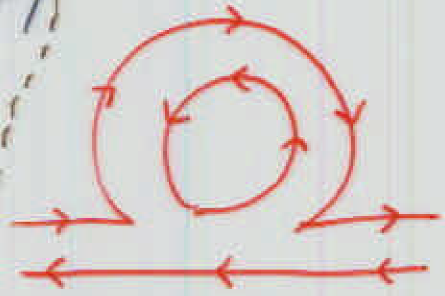
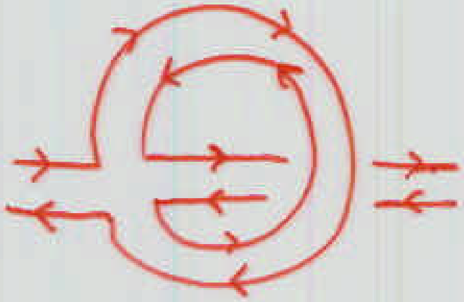
UV-limit

⇔ $\alpha \to 0$ limit

$$\sim -\frac{\lambda}{4}\frac{1}{16\pi^2}\int_0 \frac{d\alpha}{\alpha^2}\, e^{-\frac{1}{4}\frac{1}{\alpha}\tilde{p}^2}$$

: converges

(i.e.) nonplanar diagram is UV-finite in NC field theory.

NC field theory		large-N field theory
$a\,\lambda\,\Lambda_{cut}^2$		$a\,\lambda_H\,\Lambda_H^2$
finite		$b \times \frac{1}{N}\,\lambda_H\,\Lambda_H^2$

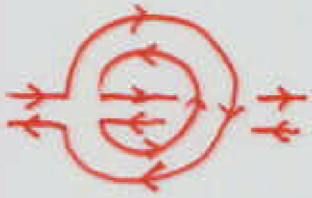
$\boxed{\text{UV-limit of NC field theory} \equiv \text{UV and planar limit of the corresponding large-N field theory}}$

(Bigatti & Susskind
Ishibashi, Iso, Kawai & Kitazawa)

aspects of UV-limit

It is determined by planar diagrams

§4. IR aspects of NC field theory



$$= -\frac{\lambda}{4}\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}\, e^{-\frac{1}{4\alpha}\tilde{p}^2} \qquad (\tilde{p}^\mu = C^{\mu\nu} p_\nu)$$

$$\alpha = \tilde{p}^2 t$$

$$= \frac{1}{\tilde{p}^2}\left(-\frac{\lambda}{4}\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\, e^{-\frac{1}{4t}}\right)$$

... diverges in IR-side ($p \to 0$) quadratically !

origin

set $C^{\mu\nu} \to 0$

integral $= \frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}$ .... that of

.... UV-divergent !

...regularized as

$$\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}\, e^{-\frac{1}{4\alpha}\frac{1}{\Lambda^2}}$$

$$= \Lambda^2\left(\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^2}\, e^{-\frac{1}{4t}}\right)$$

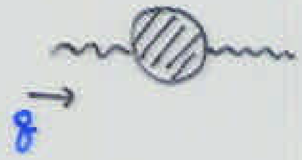
$$\frac{1}{\tilde{p}^2} \leftrightarrow \Lambda^2$$

IR-divergent behavior ($\sim 1/\tilde{p}^2$) generated by nonplanar diagram

reflects UV-divergent ($\sim p^2$) behavior of planar diagram

U(1) noncommutative Yang-Mills theory

NCYM



$\Big|_{\text{NCYM, transverse}}$

UV $\longrightarrow$ $\boxed{-\frac{g^2}{16\pi^2}\,\frac{10}{3}\,\ln q^2\left(\delta_{\mu\nu}q^2 - q_\mu q_\nu\right)}$

$|q| \gg \frac{1}{\sqrt{|C|}}$

$$Z_A = 1 - \left(-\frac{g^2}{16\pi^2}\,\frac{10}{3}\right)\ln\Lambda^2$$

IR $\longrightarrow$ $\boxed{-\frac{g^2}{16\pi^2}\,\frac{10}{3}\left[\underline{-\ln(\tilde{q}^2)}\right]\left(\delta_{\mu\nu}q^2 - q_\mu q_\nu\right)}$

$|q| \ll \frac{1}{\sqrt{|C|}}$ $\qquad$ >0 for $|q| \ll |C|^{1/2}$

- UV-singularity $\cdots$ $\ln q^2 \xleftrightarrow[\text{agree}]{} \ln \tilde{q}^2$ $\cdots$ IR-singularity

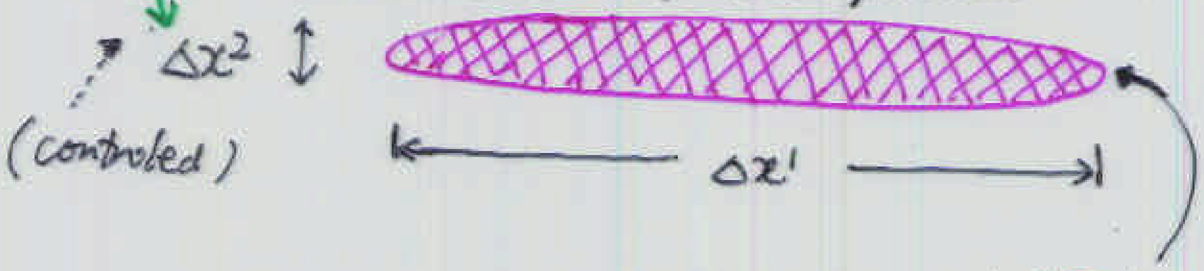
- The coefficients are also the same.

$$[X^\mu, X^\nu] = -i\, C^{\mu\nu}\, 1_N$$

$$\Delta x^\mu \Delta x^\nu \geq 2\pi |C^{\mu\nu}|$$

- Let us observe short distance phenomena in a direction

… the bulk configurations which determine the behavior of the system:



$$\Delta x^1 \Delta x^2 \geq 2\pi |C^{12}|$$

Thus, we also have to observe long distance features !!

(Note:) An example which does not give the correspondence between infrared and ultraviolet sides…. noncommutative QED

precise

↓

We cannot find

the large N field theories

associated with noncommutative QED

§5. Conclusion and discussion

We have observed a few fundamental properties of noncommutative quantum field theory:

① UV limit is governed by the planar diagrams, and (usually) also described by the corresponding large N field theory.

② A new type of singularity in the IR side is generated by the nonplanar diagrams.

It has a close relationship to UV-limiting behavior. -- "UV - IR mixing"

due to Minwalla, Raamsdonk & Seiberg

the practice issue:

whether those aspects of noncommutative field theory accomodate

the quantum theory of gravity

and the (interacting) string theory