Illusory are the conventional anomalies in the conformal-gauge two-dimensional quantum gravity

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based on the work done in collaboration with

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BRS formulation of conformal-gauge two-dimensional quantum gravity

coupled with D ($\neq 26$ in general) massless scalar fields.

Not intend to criticize non-field-theoretical approaches to string theory such as conformal field theory, path-integral approach without ghosts, etc. (Rather, those approaches are not equivalent to the genuine field-theoretical formalism based on the fundamental action involving ghosts.)

Conformal-gauge 2d QG reduces to a free field theory if B-field (= BRS daughter of FP antighost) is eliminated (or path-integrated) at the starting point. This fact is usually discarded because it contradicts the expectation from string theory (cf. Fujikawa, Kato-Ogawa).
Truth can be found by constructing explicitly exact solution of the model without eliminating B-field.

The problem is of very Deep *level.*

Basic aspects of operator-formalism approach and path-integral approach (= Feynman-diagrammatic approach)

Two essential differences:

① Operator formalism consists of two steps; operator algebra and its representation in terms of state vectors. Path-integral approach directly gives the solution without distinguishing operator level and representation level.

② Path integral can describe only the quantities which are expressed in terms of *T*-product*, while operator formalism can describe any kind of operator products.

T*-product is *different* from T-product.

T*-product is defined by the prescription that all differentiation should be made *after* taking T-product of fundamental field operators.

T-product generally violates field equations.*

Hence Noether theorem no longer holds inside T*-product. This violation of current conservation is often misidentified with current anomaly.

Amount of field-equation violation in path integral

From path-integral measure invariance under functional translation of φ:

$$\langle \mathrm{T}^*(\delta/\delta\varphi)S\cdot F\rangle - i\,\langle \mathrm{T}^*(\delta/\delta\varphi)F\rangle = 0$$

φ: generic field

S: action

F: arbitrary function of fields

Second term expresses the deviation from field equation $(\delta/\delta\varphi)S=0$.

Operator formalism derivation:. From the equivalence between the field equation and the Heisenberg equation, one has $(\delta/\delta\varphi)S = -\partial_0\pi + i\,[H,\pi]$. Substituting it into $\langle \mathrm{T}(\delta/\delta\varphi)S\cdot F\rangle = 0$ and pulling out ∂_0 outside T-product, one encounters an equal-time commutator between π and F, which yields the second term

Thus as long as canonical conjugate, π, of φ exists, the T*-product effect can be reproduced by operator formalism.

The B-field of the model, however, does *not* have its canonical conjugate. Hence path integral *can* give a result *different* from operator formalism.

How to construct the solution in the operator-formalism approach

Given a system of field equations $(\delta / \delta\varphi_A)S = 0$
and equal-time commutation relations
Rewriting field equations into *commutator form*

$$[(\delta / \delta\varphi_A(x))S,\ \varphi_B(y)] = 0,$$

one sets up *q-number Cauchy problem* for $[\varphi_A(x), \varphi_B(y)]$. Solving it, one obtains an infinite-dimensional Lie algebra for field operators. Then one constructs its matrix representation by introducing state vectors. This is realized by constructing a set of Wightman functions (=vacuum expectation values of simple products) under certain conditions characterizing the vacuum (= energy positivity and generalized normal ordering).

Crucial mathematical point to be noted here is

This matrix algebra is required to be a representation of *Lie algebra* of fields but *not* of its *universal enveloping algebra* defined by algebraic products of fields.

Original field equations *may* not completely be satisfied at representation level:

"field-equation anomaly"

Field-equation anomaly cannot be described by path integral.

Conformal-gauge 2d QG in BRS formalism

gravitational field $g_{\mu\nu}$: gauge-fixed to $\rho\eta_{\mu\nu}$, where $\rho(x)>0$

matter fields: D massless scalar fields ϕ_M ($M = 0, 1, \ldots, D-1$). Parametrizing $g_{\mu\nu}$ by a traceless symmetric tensor $h^{\mu\nu}$, one has an action *independent* of ρ. FP-ghost c^λ is a vector, while B-field $\tilde{b}_{\mu\nu}$ and FP-antighost $\bar{c}_{\mu\nu}$ are traceless symmetric tensors. But it is possible to rewrite $h^{\mu\nu}, \tilde{b}_{\mu\nu}, \bar{c}_{\mu\nu}$ into *vector-like quantities* $h_\lambda, \tilde{b}^\lambda, \bar{c}^\lambda$.

With $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, the Lagrangian density is given by

$$\pounds = [-\tfrac{1}{2}\tilde{b}^+ h_+ - i\bar{c}^+\partial_- c^+ + (+\leftrightarrow -)] + \partial_+\phi_M \partial_-\phi^M + \pounds_I$$

with

$$\pounds_I = \tfrac{1}{2} h_+[-2i\bar{c}^+\partial_+ c^+ - i(\partial_+\bar{c}^+ \cdot c^+ + \partial_-\bar{c}^+ \cdot c^-) + \partial_+\phi_M \cdot \partial_+\phi^M]) + (+\leftrightarrow -) + O(h^2)$$

$O(h^2)$: higher-order terms of h_λ

Field equations:

$$h_\pm = 0,$$
$$\tilde{b}^\pm = \tfrac{1}{2}\,\delta\,\pounds_I / \delta\, h_\pm, \qquad \Rightarrow \quad \partial_\mp \tilde{b}^\pm = 0$$
$$\partial_\mp c^\pm = 0, \qquad \partial_\mp \bar{c}^\pm = 0,$$
$$\partial_+\partial_-\phi_M = 0$$

Any of $\tilde{b}^\pm, c^\pm, \bar{c}^\pm, \partial_\pm\phi_M$ is a function of $x^\pm$ *alone*.

This remarkable simplicity is valid only in operator formalism, but *not* in path-integral formalism because of the use of T*-product.

Exact solution in terms of Wightman functions

Nonvanishing truncated Wightman functions:
n-point functions consisting of
either ϕ_M, ϕ_N and $n-2$ B-fields
or $c^{\pm}$, $\bar{c}^{\pm}$ and $n-2$ B-fields

Thus if B-field is not considered, the solution is a free-field one.

Exact solution is completely consistent with BRS invariance and FP-ghost number conservation.

Subtlety arises for $\tilde{b}^{\pm} = \frac{1}{2}\,\delta\,£_I / \delta\, h_{\pm}$

It is slightly (i.e., modulo $\partial_{\mp}\tilde{b}^{\pm}=0$) violated at representation level; that is, it suffers from *field-equation anomaly*:

$$\langle \tilde{b}^{\pm}(x_1)[\tilde{b}^{\pm}(x_2) - \tfrac{1}{2}\,\delta\,£_I/\delta\, h_{\pm}(x_2)]\rangle = -2(D-26)\left[\partial_{\pm}^{\,2} D^{(+)}(x_1-x_2)\right]^2$$

Feynman-diagrammatic calculation

Results become very ***crazy*** *owing to T*-product's field- equation violation*

(1.) In addition to $\langle T^*\bar{c}^{\pm}(x_1)c^{\pm}(x_2)\rangle$ and $\langle T^*\phi_M(x_1)\phi_N(x_2)\rangle$, there is a nonvanishing 2-point function

$$\langle T^*\tilde{b}^{\pm}(x_1)h_{\pm}(x_2)\rangle = -2i\delta^2(x_1-x_2)$$

in apparent contradiction with $h_{\pm}=0$

It induces many pathological effects which are absent in operator formalism.

Feynman rules based on $\pounds_I$ imply existence of one-loop diagrams for *n*-point function consisting of B-fields only; e.g.,

$$\langle T^*\tilde{b}^{+}(x_1)\tilde{b}^{+}(x_2)\rangle = 2(D-26)\left[\partial_+^2 D_F(x_1-x_2)\right]^2 \qquad (*)$$

$$\langle T^*\tilde{b}^{+}(x_1)\tilde{b}^{-}(x_2)\rangle = -½(D-2)\left[\delta^2(x_1-x_2)\right]^2 \qquad (\#)$$

(*) shows *BRS anomaly appears for* $D\neq 26$.

(2.) T*-product does not respect the fact that any of $\tilde{b}^{\pm}, c^{\pm}, \bar{c}^{\pm}, \partial_{\pm}\phi_M$ is a function of $x^{\pm}$ *alone*. Therefore, the Green's functions consisting of *both* + components *and* − components can remain ***nonvanishing***; e.g., (#) and

$$\langle T^*c^{+}(x_1)\tilde{b}^{+}(x_2)\bar{c}^{-}(x_3)\rangle = -2\delta^2(x_1-x_2)\partial_- D_F(x_2-x_3)$$

Decoupling of right-moving and left-moving is not realized in path integral !!

(3.) An infinite number of higher-order terms $O(h^2)$ in $\pounds_I$ ***do contribute*** to the higher-point functions containing more than one B-fields.

BRS Noether current

$$j_b^{\mp} = j_b'^{\mp} + (\widetilde{b}^{\pm} - ½\, \delta\, £_I / \delta\, h_{\pm})c^{\pm}$$

with

$$j_b'^{\mp} \equiv -\widetilde{b}^{\pm} c^{\pm} + i\overline{c}^{\pm} c^{\pm} \partial_{\pm} c^{\pm}$$

If one defines BRS charge by using BRS *Noether* current, one encounters anomaly for $D \neq 26$. This is nothing but the result found by Kato and Ogawa.

(Finiteness effect of string length can be taken into account without bringing any essential change into the conclusion.)

But if BRS charge is defined by using $j_b'^{\mp}$, which, is completely equal to $j_b^{\mp}$ at operator level, then *there is no anomaly even for* $D \neq 26$.

Thus violation of BRS-charge nilpotency for $D \neq 26$, claimed by Kato and Ogawa, is *not an intrinsic result* but a consequence of unconsciously taking in field-equation anomaly.

FP-ghost number Noether current

$$j_c^{\mp} = -i\bar{c}^{\pm} c^{\pm}$$

It conservation law follows from $\partial_{\mp} c^{\pm} = 0$ and $\partial_{\mp} \bar{c}^{\pm} = 0$.

The "anomaly" is implied by the Feynman-diagrammatic result

$$\langle \mathrm{T}^* j_c^{\mp}(x_1) \text{½} \, \delta \, \text{£}_I / \delta h_{\pm}(x_2) \rangle$$
$$= -3\partial_{\pm} D_F(x_1 - x_2) \cdot \partial_{\pm}^2 D_F(x_1 - x_2)$$

Its violation of conservation law is merely due to the use of T*-product.

Indeed, without T*, D_F is replaced by $D^{(+)}$ in r.h.s., so that conservation law of $j_c^{\mp}$ is perfectly all right.

Thus ***FP-ghost number current anomaly an illusion caused by T*-product.***

(One might say that the existence of FP-ghost number current anomaly is a consequence of Riemann-Roch theorem. This assertion is wrong because this theorem holds only in *global* sense; locally, only one additional point to spacetime manifold can change the conclusion of the theorem.)

Conclusion

1. Owing to the use of T*-product, Feynman-diagrammatic or path-integral calculation yields very crazy results in BRS formulation of conformal-gauge two-dimensional quantum gravity.

2. Kato-Ogawa's violation of BRS-charge nilpotency for $D \neq 26$ is not an intrinsic result. BRS invariance is not violated for any value of D.

3. FP-ghost number is conserved completely; "FP-ghost number current anomaly" is an illusion caused by T*-product.

Similar misleading discussions are found concerning Virasoro anomaly and gravitational anomaly in a book by Green, Schwarz and Witten ("*Superstring Theory: 1*" pp.141-142 and pp.145-146).