Can CPT be Violated Through Extended Time Reversal

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Why to Consider Possible Extensions of Time Reversal?

i) Direct time reversal is experimentally not studied enough. So it is not clear yet if CPT can be violated at the elementary level.

ii) Even if CPT is conserved there may be still particles with unusual time reversal properties, e.g. Wigner types

In Which Ways Can Time Reversal be Extended?

There are two ways to extend time reversal:

i) By using the method of group extensions, which amounts to associating a internal group in direct product with Poincare group. The simplest analog is the situation of parity in the standard model of electroweak interactions.

ii) By considering the doubling of the Hilbert space in projective representation of time reversal due to the antiunitarity of time reversal, which is first studied by Wigner and extended later S. Weinberg.

Of course one can also consider a combination of the two alternatives given above.

What is our contribution?

i) We have clarified the physical content of the extended time reversal, especially by embeding the scheme into higher dimensional spaces;

ii) furher studied these ideas more explicitly through a toy model;

iii) pointed out the possibility of violation of CPT in the context of field theory by using extended time reversal.

Extension of Time Reversal Through Wigner Types and Group Extensions

The symmetry transformations in quantum theory, $U(g)$ are defined up to a phase factor

$$U(g) \equiv e^{i\alpha(g)} U(g) \quad g \in G \text{ (for some group G)} \qquad (1)$$

because physical states are described by rays rather than vectors in the corresponding Hilbert space. So the general possible representation of G in quantum theory is a projective representation where

$$U(g_1)U(g_2) = e^{i\phi(g_1,g_2)} U(g_1 g_2) \quad g_1, g_2 \in G \qquad (2)$$

Wigner Noticed that if one uses projective representation of time reversal there two types of time reversal operators:

i)$\mathcal{T}^2 = -1$

ii)$\mathcal{T}^2 = 1$

since $\mathcal{T}^2 = \omega I$ (where ω is some phase) because of Eq.(2), hence (through associativity property of $\mathcal{T}$)

$$\mathcal{T}^2\mathcal{T} = \omega\mathcal{T} = \mathcal{T}\mathcal{T}^2 = \omega^*\mathcal{T} \tag{3}$$

These possibilities can be realized through the following representations

$$\mathcal{T} = \begin{pmatrix} 0 & 1\,U(T)\,K \\ 1\,U(T)\,K & 0 \end{pmatrix} \quad , \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} 0 & i\,U(T)\,K \\ -i\,U(T)\,K & 0 \end{pmatrix}$$

respectively, where $U(T)$ is some unitary matrix and K stands for the antilinear part of $\mathcal{T}$, which takes complex conjugate of c-numbers.

S. Weinberg has given a generalization of these possibilities by combining them in one general operator, that is,

$$\mathcal{T} == \begin{pmatrix} 0 & e^{i\frac{\phi}{2}}U(T)K \\ e^{-i\frac{\phi}{2}}U(T)K & 0 \end{pmatrix} \tag{5}$$

For spin 1/2 fields U(T) is a 4×4 matrix. So the minumum dimension where the unusual Wigner types can arise is an 8-component spinor representation.

Once we have adopted 8-component spinors we have further possibilities for time reversal in addtion to those provided by Wigner types. One can assign the upper and lower 4-component parts of the 8-component spinor to different representations of internal group. Furthermore one can take the upper and lower to be related by time reversal as in the case 4-compoent spinor and parity. This offers us another way to extend the usual time reversal.

8-component spinor naturally arise in spinor representation of space-times with dimensions higher than 5. For example consider a 6 dimensional Minkowski vector, with the metric $(g_{AB})=(1,-1,-1,-1,-1,-1)$, embeded in the corresponding Clifford algebra

$$X = \Gamma^A x_A = \begin{pmatrix} V & ix_6\gamma^0 \\ ix_6\gamma^0 & -\gamma^0 V \gamma^0 \end{pmatrix}, \quad V = i\gamma_5 + \begin{pmatrix} 0 & \omega \\ \omega' & 0 \end{pmatrix}$$

$$\omega = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad \omega' = \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix} \tag{6}$$

We indentify x_0 by time. Then time reversal, $X^{(T)}$ of X is

$$X^{(T)} = \begin{pmatrix} -\gamma^0 V \gamma^0 & ix_6\gamma^0 \\ ix_6\gamma^0 & V \end{pmatrix} = \Gamma^0 X \Gamma^{0\dagger} \tag{7}$$

where either

$$\Gamma^0 = \begin{pmatrix} i\gamma^1\gamma^3 J & 0 \\ 0 & i\gamma^1\gamma^3 J \end{pmatrix} \quad \text{or}$$

$$\Gamma^0 = \begin{pmatrix} 0 & i\gamma^2 J \\ -i\gamma^2 J & 0 \end{pmatrix} \quad \text{or} \quad \Gamma^0 = \begin{pmatrix} 0 & \gamma^2 J \\ -\gamma^2 J & 0 \end{pmatrix}$$

up to a phase. Here J stands for the antiunitary part of time reversal, which takes the complex conjugate of c-numbers.

The first time reversal operator corresponds to the usual time reversal operator. The second time reversal operator corresponds to the unusual Wigner types with $\mathcal{T}^2 = 1$ and the last one corresponds to the usual Wigner types with $\mathcal{T}^2 = -1$ accomponied with a reversal of the upper and lower components of the 8-component spinor. If the upper and lower components are assigned to different representations of the internal group then the last two time reversal operators may include a transformation in the internal space as in the case of parity. Hence the following time reversal (where the time reversal causes a spatial inversion in the sixth direction) is also allowed

$$\Gamma^0 = \begin{pmatrix} 0 & \gamma^2 J \\ \gamma^2 J & 0 \end{pmatrix} \tag{9}$$

Check of One Particle Results for Field Theory

We assume that 8-component spinors arise from the Clifford algebra of a higher dimensional space-time or the physical outcome of the underlying theory is equivalent to this formulation. So particle species are localized in the extra space-time coordinates and these coordinates determine the internal quantum numbers of the particles. Then the general wave expansion of a solution of the 8-dimensional free Dirac equation, in terms of creation and annihilation operators, is

$$\begin{aligned} \psi(\vec{x},t) &= \sum_{\pm s}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}[b(p,s)\,u(p,s)\,e^{+ip.x} \\ &+d^{\dagger}(p,s)\,v(p,s)\,e^{ip.x}] \qquad (10) \\ &\text{where } p.x = Et - \vec{p}.\vec{x}\ ,\quad p.p = m^2 \end{aligned}$$

The form of the 8-component fermion field is essentially the same as the 4-component fermion field except u and v in this case are

8-component spinors

$$u(v) = \begin{pmatrix} u_1(v_1) \\ u_2(v_2) \end{pmatrix} \tag{11}$$

where $u_1(v_1)$, $u_2(v_2)$ are 4-component spinors.

Next we investigate if these time reversal operators obtained for one particle theory apply in field formalism as well in a consistent way. Similar to 4-component case under time reversal (T), ψ transforms as

$$\begin{aligned} & \mathcal{T} : \Psi(\vec{x}, t) \rightarrow \mathcal{J}\Psi\mathcal{J}^{-1} = U(\mathcal{T})\Psi(\vec{x}, -t) \\ = \quad & \sum_{\pm s} \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E_p}} [\mathcal{U} b(p, s) \mathcal{U}^{-1} u^*(p, s)\, e^{ip.x} \\ & + \mathcal{U} d^\dagger(p, s) \mathcal{U}^{-1} v^*(p, s)\, e^{ip.x}] \\ = \quad & \sum_{\pm s} \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E_p}} b(p, s), u^*(-p, -s)\, e^{i(Et+\vec{p}.\vec{x})} \\ & + d^\dagger(p, s)\, v^*(-p, -s)\, e^{ip.x}] \quad , \quad p.p = m^2 \end{aligned} \tag{12}$$

where $\mathcal{J}$ is an antiunitary operator standing for time reversal and $\mathcal{U}$ denotes its unitary part. We have used

$$\mathcal{U}b(\vec{p},s)\mathcal{U}^{-1} = b(-\vec{p},-s)\ ,\quad \mathcal{U}d(\vec{p},s)\mathcal{U}^{-1} = d(-\vec{p},-s) \tag{13}$$

and changed variables $\vec{p}$, s to their minuses in Eq.(12). The $u^*(-\vec{p},-s)$ and $v^*(-\vec{p},-s)$ in (12) can be related to $u(\vec{p},s)$ and $v(\vec{p},s)$ in the usual way

$$u(-\vec{p},-s) = D(L(-\vec{p}))u(0,-s)\ ,\quad v(-\vec{p},-s) = D(L(-\vec{p}))v(0,-s) \tag{14}$$

where $D(L(\vec{p})$ stands for the representation of Lorentz boost (from zero momentum to $\vec{p}$) corresponding to Ψ. For an n dimensional space D is given by

$$D(L(\vec{p})) = \quad exp(\frac{i}{2}\omega_{0k}J^{0k})\ ,\quad k=1,2,3 \tag{15}$$

$$J^{0k} = -\frac{i}{4}[\Gamma^0,\Gamma^k]$$

ω_{ok} is the boost parameter

One can explicitly check that for the time reversal operators in Eq.(8) and Eq.(9) the following identities are true

$$D^*(L(-\vec{p}) = \Gamma^0 D(L(\vec{p})\Gamma^{0\dagger} \tag{16}$$

$$\Gamma^0\, u^*(0,-s) = (-1)^{\frac{1}{2}-s} u_p(0,s) \tag{17}$$

where the subscript p stands for the fact that the extra space-time dimension dependence (which corresponds to internal space dependence in our identification) of u, in general, may change after time reversal. The last identity directly follows from the fact that $u_2(v_2)$ is the time reversal of $u_1(v_1)$ in Eq.(11). After using Eq.(16) in Eq.(12) one arrives to the same transformation rule as one particle theory.

A Toy Model

In the light of the above discussion and the possibility of doubling the Hilbert space due to the anti-unitary nature of time reversal we shall give a realization of the theoretical framework for extended time reversal discused above. We shall discuss some of its implications in this section and some others in the following section. There are three main routes one can follow for this purpose:

i) One can couple some (usual) fermions to (usual) time revesal of some other (usual) fermions provided each set has different internal group representations. In this case the extended time reversal amounts to the usual time reversal followed by a Z_2 internal space transformation. In other words we extend the definition of time reversal as the usual time reversal followed by an intrinsic time reversal transformation (the intrinsic time reversal degree of freedom can be identified, for example, with the different transformation properties of ψ_1 and ψ_2 under the internal group as in the standard model in the case of parity), that is,

$$\mathcal{T}:\psi_{1(2)}(p_{1(2)}) \to T:\psi_{2(1)}(p_{1(2)})$$

$$\psi_{1(2)}(\vec{x},t) \sim T:\psi_{2(1)}(\vec{x},-t) \qquad (18)$$

The second line of Eq.(18) effectively means that the spinor part of $\psi_{2(1)}$ behaves as the time reversed of $\psi_{1(2)}$. We couple ψ_1, ψ_2 through the following interaction

$$\mathcal{L} = m\bar{\psi}_1\psi_1 + m\bar{\psi}_2\psi_2 + M\psi_1^\dagger\psi_2 + M\psi_2^\dagger\psi_1$$

$$= \Psi^\dagger M'\Psi \qquad (19)$$

$$M' = \begin{pmatrix} \gamma^0 m & M \\ M & \gamma^0 m \end{pmatrix} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad (20)$$

After the inclusion of the fermion kinetic term the Lagrangian becomes

$$\mathcal{L} = \bar{\Psi}D_\mu\Gamma^\mu\Psi + \bar{\Psi}\tilde{M}\Psi \qquad (21)$$

$$\tilde{M} = \begin{pmatrix} m & \gamma^0 M \\ \gamma^0 M & m \end{pmatrix} \quad \Gamma^\mu == \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \tag{22}$$

The lagrangain Eq.(19) introduces no time additional time reversal violation other than the one in the 4-component case through CP violation. However if we change Eq.(19) into

$$\begin{aligned} \mathcal{L} &= m\bar{\psi}_1\psi_1 + m'\bar{\psi}_2\psi_2 + M\psi_1^\dagger\psi_2 + M\psi_2^\dagger\psi_1 \\ &= \Psi^\dagger M'\Psi \end{aligned} \tag{23}$$

there will be an additional source of time reversal for $m' \neq m$. One can suppress this violation by taking $|m - m'| << m$. The physical content of the model can be seen better by the diagonalization of the corresponding $\tilde{M}$ which results in two 4-component spinors ψ, ψ^c,

$$\psi \propto m''\psi_1 - M\gamma^0\psi_2 \, , \quad \psi^c \propto M\gamma^0\psi_1 + m''\psi_2$$

$$m'' = \frac{1}{2}(m - m' + \sqrt{(m + m')^2 - 4mm' + 4M^2})$$

So the physical fermions are the mixtures of a set of fermions with another set of fermions whose electric charges are opposite to the original set. If one identifies ψ_1 with usual particles then the conservation of electric charge allows the mixing in Eq.(24) only for neutrinos.

ii) The second option is to take the general form of time reversal while assuming that the particle does not change its internal group representation under the time reversal. We re-express the extended time reversal, $\mathcal{T}$ as

$$\mathcal{T} = \begin{pmatrix} 0 & e^{i\eta} U(T) \\ e^{-i\eta} U(T) & 0 \end{pmatrix} \tag{25}$$

The above form of $\mathcal{T}$ which is introduced by S. Weinberg includes the two cases, $\mathcal{T}^2 = 1$ and $\mathcal{T}^2 = -1$ as subcases. $\mathcal{T}^2 = 1$ and

$\mathcal{T}^2 = -1$ correspond to $e^{i\eta} = 1$ and $e^{i\eta} = i$, respectively. Although the operator $\mathcal{T}^2$ generates superselection rules for $\mathcal{T}^2 = \pm 1$ (becasue it commutes with all operators) This is not true for arbitrary value of η because in that case $\mathcal{T}^2$ is not proportional with unit matrix in general. This allows mixing of the particle states belonging to the Hilbert spaces $\mathcal{T}^2 = -1$ and $\mathcal{T}^2 = 1$,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \cos\eta \begin{pmatrix} e^{-i\frac{\eta}{2}}\psi_1^{(1)} \\ e^{i\frac{\eta}{2}}\psi_2^{(1)} \end{pmatrix} + \sin\eta \begin{pmatrix} e^{-i(\frac{\eta}{2}+\frac{\pi}{4})}\psi_1^{(-1)} \\ e^{i(\frac{\eta}{2}+\frac{\pi}{4})}\psi_2^{(-1)} \end{pmatrix} \tag{26}$$

where the superscripts (1) and (−1) denote the $\mathcal{T}^2 = -1$ and $\mathcal{T}^2 = 1$ subspaces, respectively. This mixing, in turn, makes it impossible to absorb the phase η into redefinition of ψ_1 and ψ_2. However in this case the Lagrangian (19) is not invariant under time reversal. One can suppress the time reversal violation by taking $M << m$ in this case. One should also take $\eta \simeq 0$ in order to remain in agreement with experimental data which has no evidence for the unusual fermions with $\mathcal{T}^2 = 1$. Moreover one should take $|m - m'| << m$ (and $M << m$) so as not to run into conflict with experimental data which has no evidence for additional light fermions.

iii) The last option is to take the extended time reversal and furthermore to assign the upper and the lower components of the 8-component spinor to different internal group representations. This is similiar to the previous option in its bevaiour for the Lagrangian in (19) but there may be an additional time reversal violation in the gauge sector due to the noninvaraince of time reversal under the internal group transformations. The present option is the most interesting one because one can assign $\psi_1$to the standard model gauge group while taking ψ_2 to transform under a different gauge group. In this way one does not need to impose the condition $|m - m'| << m$.

Hence one can consider ψ_2 as a good candidate for sterile neutrinos. Another interesting aspect of this option is that the validity of CPT theorem in this case is not clear as we shall see in the next section.

8-component formulation and CPT

The Lagrangian in Eq.(19) is invariant under the interchange of the subscripts 1 and 2. One can also take the other terms of the full Lagrangian invariant under this interchange. In this case T transformation will be equivalent to time reversal so that CPT theorem necessarily holds. In other words CPT theorem is always true in the case i) of the previous section. However one can make the Lagrangian non-invariant under this interchange, for example, by taking the mass term in front of $\bar{\psi}_1\psi_1$ to be different than the one in front of $\bar{\psi}_2\psi_2$

$$\mathcal{L} = m\bar{\psi}_1\psi_1 + m'\bar{\psi}_2\psi_2 + M\psi_1^\dagger\psi_2 + M\psi_2^\dagger\psi_1 = \Psi^\dagger M'\Psi \quad (27)$$

or one can take the general time reversal operator introduced by S. Weinberg (which corresponds to the mixture of two Wigner types to make the applicablity of CPT theorem questionable or one can take the case iii) in the previous section, that is, both we adopt the extended time reversal and we assign the upper and the lower 4-components of the 8-component spinor to different internal group representations. In these cases CPT invariance is not automatic as we shall see below.

CPT theorem states that the condition of weak local commutivity for fields (which is satisfied by all reasonable fields) is enough for the following equation to be satisfied

$$(\Psi_0, \phi_\mu(x_1)\phi_\nu(x_2)\ldots\ldots\phi_\rho(x_n)\Psi_0)$$
$$= i^F(-1)^J(\Psi_0, \phi_\mu(-x_1)\phi_\nu(-x_2)\ldots\ldots\phi_\rho(-x_n)\Psi_0) \quad (28)$$

(where F stands for the number of Majorana type fermions and J is the angular momentum of the (fermion* bilinear) ϕ's).

Provided they satisfy the following axioms

[redacted]

I. For each test function f defined on space-time, there exists a set of $\phi_1(f), \phi_2(f), ..., \phi_n(f)$ and their adjoints which are defined on a domain D of vectors dense in the Hilbert space, $\mathcal{H}$. Furthermore D is a linear space containing Ψ_0. All the vectors obtained after the application of (i- the Lorentz group transformations, ii- the field operators) are in the physical Hilbert space, $\mathcal{H}$. Moreover for all $\Phi, \Psi \in D \subset \mathcal{H}$ and $\phi_\mu(f)$ being a field defined as a functional of the test function f, $(\Phi, \phi_\mu(f)\Psi)$ is a tempered distribution.

II. The equation

$$U(\Lambda, a)\phi_j(f)U(\Lambda, a)^{-1} = \sum S_{jk}(a^{-1})\phi_k(\{\Lambda, a\}f) \quad (29)$$

is satisfied; here $\{\Lambda, a\}f = f(a^{-1}(x - a))$ and $U(\Lambda, a)$ is the unitary representation of Poincare group.

In the case of 4-component spinors CPT invariance reduces to the validity of Eq.(28). This can be seen as follows: Under the discrete space-time transformations the usual fermions transform as

$$
\begin{aligned}
&P : \psi_{1(2)}(\vec{x},t) \to e^{i\beta_P}\gamma^0\psi_{1(2)}(-\vec{x},t) \\
&C : \psi_{1(2)}(\vec{x},t) \to ie^{i\beta_C}\gamma^2\gamma^0\bar{\psi}^T_{1(2)}(\vec{x},t) \\
&T : \psi_{1(2)}(\vec{x},t) \to ie^{i\beta_T}\gamma^1\gamma^3\psi_{1(2)}(\vec{x},-t) \\
&CPT : \psi_{1(2)}(\vec{x},t) \to -i\,e^{i(\beta_C-\beta_P-\beta_T)}\gamma^0\gamma_5\bar{\psi}^T(-\vec{x},-t)
\end{aligned} \tag{30}
$$

here ψ denotes the whole fermionic field (not just the spinor part on the contrary to the previous section) and the subscripts 1(2) denote the upper and lower 4-component parts of the 8-component spinor.The fermion bilinear $X^{AB} = \bar{\psi}^A\Gamma\psi^A$, under CPT, transforms as

$$CPT : \bar{\psi}^A\Gamma\psi^B \to \bar{\psi}^B\Gamma'\psi^A \tag{31}$$

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$$\Gamma' = \Gamma \quad \text{for} \quad \Gamma = 1, i\gamma_5, \sigma_{\mu\nu} = [\gamma_\mu, \Gamma_\nu]$$

$$\Gamma' = -\Gamma \quad \text{for} \quad \gamma_\mu, \gamma_5\gamma_\mu$$

where the superscripts A, B denote different species of fermions. After contraction of X^{BA} with itself or with other X^{AB}'s and comparing it with Hermitian conjugate of the original terms result in Eq.(28). One can perform a similiar and simpler procedure for the other type of fields (i.e. for scalars, vector fields, etc.) to get the same conclusion. So in the usual 4-component spinor case the validity of Eq.(28) is equivalent to CPT invariance.

On the other hand in the case of extended-T this is not the case. For example take the second operator in Eq.(8) as the relavant extended time reversal operator

$$\Gamma^0 = \begin{pmatrix} 0 & i\gamma^2 J \\ -i\gamma^2 J & 0 \end{pmatrix} \tag{32}$$

while all other discrete space-time transformation remain the same time reversal and CPT become

$$
\begin{aligned}
&T : \psi_{1(2)}(\vec{x},t) \rightarrow -(+) i e^{i\beta_T}\gamma^2 \psi_{2(1)}(\vec{x},-t)\\
&CPT : \Psi(\vec{x},t) \rightarrow i L^0 e^{i(\alpha_P+\alpha_C+\alpha_T)}\gamma^0 \bar{\Psi}^T(-\vec{x},-t)\\
&L^0 = \begin{pmatrix} 0 & e^{i\eta} I_4 \\ e^{-i\eta} I_4 & 0 \end{pmatrix}
\end{aligned}
\tag{33}
$$

One can see from the above derivation that CPT invariance is not equivalent to the validity of Eq.(28) in this case. For example

$$
CPT : \bar{\Psi}^A \tilde{M} \Psi^B \rightarrow -\bar{\Psi}^B L^0 \tilde{M} L^0 \Psi^A \tag{34}
$$

So the requirement of the equality of the Hermitian conjugate of Eq.(34) to the original term does not imply Eq.(28). If one takes T as the usual time reversal one obtains Eq.(28) but then the hypothesis II., Eq.(29) is not satisfied because ψ_1 mixes with ψ_2 so ψ_1 does not span the whole physical Hilbert space

One can understand the above conclusion through a more physical argument as follows:

i) In the case of of extension by Wigner types the mixture of different types causes mixture of $\mathcal{T}^2$ odd and even terms in fermion bilinears.

ii) In the case of extension by group extensions $\mathcal{T}$ causes a rotation in the internal space while there is no additional transformation similar to C to compansate this rotation.

(After one introduces an intrinsic parity a PT transformation (on the spinor part of the fermion field) in the spinor representation of the Lorentz group is not equivalent to identity because P is not the simple space inversion anymore. However one can introduce the charge conjugation, C so that CP is effectiveley equivalent to space inversion so that CPT on the spinor part of the fermion field is equivalent to identity.

Conclusions

i) One can give an extended form of time reversal in space-times higher than 5 by using either extending time reversal by non-trivial internal group representations or by Wigner types.

ii) It is possible to violate CPT in this scheme without going outside of field theory.

A

In this appendix we shall show that the terms with the coefficients M in the Lagrangian of Eq.(19) are really Lorentz invariant although they seem to be Lorentz non-invariant at first sight.

More explicitly by the second line of Eq.(18) we mean that we have two 4-component fermion fields ψ_1 and ψ'_2, which transform under the proper homogenous Lorentz group as usual

$$\psi_1 \to A\psi_1\ ,\ \ \psi'_2 \to A\psi'_2 \qquad (35)$$
$$A = \begin{pmatrix} s_L & 0 \\ 0 & s_R \end{pmatrix}$$

where s_L, s_R are the $SL(2,C)$ transformations given in Eq.(??). We define ψ_2 to be similar to the usual time reversed of ψ'_2, that is,

$$\psi_2(\vec{x},t) = T\psi'_2(\vec{x},-t) \qquad (36)$$

In fact this is what we mean by the second line of Eq.(18).

By using Eq.(A1), Eq.(A2), and antiunitarity of T the transformation rule of ψ_2 can be determined

$$\psi_2(\vec{x},t) \to T\ (A\psi'_2(\vec{x},-t)) = i\gamma^1\gamma^3\,A^*\psi'_2(\vec{x},t) = i\gamma^1\gamma^3\,A^*(-i\gamma^3\gamma^1)\,(i\gamma^1\gamma^3)\,\psi'_2(\vec{x},t)$$
$$= i\gamma^1\gamma^3\,A^*(-i\gamma^3\gamma^1)\,T(\psi'_2(\vec{x},-t)) = \gamma^1\gamma^3\,A^*\gamma^3\gamma^1\,\psi_2(\vec{x},t)) \qquad (37)$$

Next we use the following identities

$$\gamma^1\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2\gamma^0 = -i\gamma_5\gamma^2\gamma^0 = i\gamma_5\gamma^0\gamma^2$$
$$\gamma^3\gamma^1 = i\gamma^2\gamma^0\gamma_5 \tag{38}$$

and

$$\gamma^2 A^*\gamma^2 = -A \tag{39}$$

which can be shown explicitly by using the definition of s_L, s_R

and (most easily) by the following representation for γ^2

matrix

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{40}$$

So Eq.(A3) becomes

$$\psi_2(\vec{x},t) \to \gamma^1\gamma^3 A^*\gamma^3\gamma^1 \psi_2(\vec{x},t) = \gamma_5\gamma^0 A\gamma^0\gamma_5\psi_2(\vec{x},t) = \begin{pmatrix} s_R & 0 \\ 0 & s_L \end{pmatrix}\psi_2(\vec{x},t) \tag{41}$$

where we have used the following representation for the gamma matrices as a convenient set

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{42}$$

Therefore the terms $\psi_1^\dagger\psi_2$ and $\psi_2^\dagger\psi_1$ are Lorentz invariant under the proper homogeneous Lorentz group transformations because s_L and s_R match in the following form $s_L^\dagger s_R = I$ and $s_R^\dagger s_L = I$. The result is valid for any representation of gamma matrices because different representations of gamma matrices are related by unitary transformations. The terms $\psi_1^\dagger\psi_2$ and $\psi_2^\dagger\psi_1$ are also invariant under translations because the exponential factors in the field expansion of both ψ_1 and ψ_2 contain the same space-time dependences. A similiar conclusion is valid for parity

$$P : \psi_2(\vec{x},t) \rightarrow T\left(P\psi_2'(\vec{x},-t)\right) = T\left(\gamma^0\psi_2'(-\vec{x},-t)\right) = i\gamma^1\gamma^3\gamma^0\psi_2'(-\vec{x},t) = \gamma^0 T\psi_2'(-\vec{x},-t) = \gamma^0\psi_2(-\vec{x},t \tag{43}$$

So

$$P : \psi_{1(2)}^\dagger(\vec{x},t)\psi_{2(1)}(\vec{x},t) = \psi_{1(2)}^\dagger(-\vec{x},t)\gamma^0\gamma^0\psi_{2(1)}(-\vec{x},t) = \psi_{1(2)}^\dagger(\vec{x},t)\psi_{2(1)}(\vec{x},t) \tag{44}$$

This is also the case for the charge conjugation

$$C : \psi_2(\vec{x},t) \rightarrow T\left(C\psi_2'(\vec{x},-t)\right) = T\left(i\gamma^2\psi_2'^*(\vec{x},-t)\right) = -\gamma^1\gamma^3\gamma^2\psi_2'^*(\vec{x},t) = -i\gamma^2 T\psi_2'^*(\vec{x},-t) = -i\gamma^2\psi_2^*(\vec{x} \tag{45}$$

while ψ_1 transforms as $C : \psi_1(\vec{x},t) = i\gamma^2\psi_1^*(\vec{x},t)$. So

$$\begin{aligned} C : \psi_{1(2)}^\dagger(\vec{x},t)\psi_{2(1)}(\vec{x},t) + h.c. &= -\psi_{1(2)}^T(\vec{x},t)\gamma^2\gamma^2\psi_{2(1)}^*(\vec{x},t) + h.c. = \psi_{1(2)}^T(\vec{x},t)\psi_{2(1)}^*(\vec{x},t) + h.c. \\ &= \psi_{1(2)}^\dagger(\vec{x},t)\psi_{2(1)}(\vec{x},t) + h.c. \end{aligned} \tag{46}$$

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