Auxiliary Field Formulation of Supersymmetric Nonlinear Sigma Models

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§1. Introduction

2D Nonlinear Sigma Models (NLσM) have been interested since they have many similarities to 4D QCD.

asymptotic freedom, instanton, $\cdots$

Non-perturbative Analysis of 2D Nonlinear Sigma Models (NLσM) in Large N

$$Z = \int [d\varphi] \exp i S_{\mathrm{NL}\sigma\mathrm{M}}(\varphi),$$

φ: Nonlinear Field, $S_{\mathrm{NL}\sigma\mathrm{M}} = \frac{1}{2} g_{ij}(\varphi) \partial\varphi^i \partial\varphi^j$

$$= \int [d\phi d\sigma] \exp i \tilde{S}_{\mathrm{NL}\sigma\mathrm{M}}(\phi, \sigma),$$

σ: Auxiliary Field, ϕ: Linearized Field

$$= \int [d\sigma] \exp i S_{\mathrm{eff.}}(\sigma)$$

$$\Downarrow \; S_{\mathrm{eff.}} \sim N \; (\# \text{ of } \phi)$$

$$Z \simeq \exp i S_{\mathrm{eff.}}(\sigma_0) \quad (\text{in the large } N)$$

σ_0 is stationary points:

$$\frac{\delta S_{\mathrm{eff.}}}{\delta \sigma}\Big|_{\sigma=\sigma_0} = 0 \quad \text{Gap Equation}$$

($1/N$ expansion=loop expansion of σ)

Ex.) $O(N)$ model

Dynamical (Linear) Field: $\vec{\phi} \in \mathbf{N}$

Auxiliary Field: $\sigma \in \mathbf{1}$ of $O(N)$

$$\mathcal{L} = \partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + \sigma(\vec{\phi}^2 - a^2)$$

$\Longrightarrow$ $O(N)$ symmetry restortion

(dynamical mass generation)

(cf. Theorem of Coleman-Mermin-Wagner)

> The auxiliary field formulation is necessary for the non-perturbative analysis !

In the literature

- non-SUSY NLσM, $O(N)$ model, $\mathbf{C}P^N$ model $\rightarrow$ Dynamical Gauge Boson.
- $\mathcal{N}=1$ SUSY NLσM, $O(N)$ model (= hybrid of non-SUSY $O(N)$ model + Gross-Neveu model) $\rightarrow$ Dynamical χSB
  $\mathbf{C}P^N$ model, Grassmann model etc.

In both the theories, many models are known.

What about the $\mathcal{N}=2$ SUSY NLσM?

- $\mathcal{N}=2$ SUSY NLσM, Except for $\mathbf{C}P^N$ and Grassmann models, there was no auxiliary field formulation.
  (target space = Kähler manifold)

$$\Downarrow$$

Our new results are

Auxiliary Field Formulation of 2D, $\mathcal{N}=2$ SNLσM on Hermitian Symmetric Spaces (HSS)

HSS $\subset$ Kähler coset G/H

Type	G/H
AIII_1	$\mathbf{C}P^{N-1} = SU(N)/SU(N-1)\times U(1)$
AIII_2	$G_{N,M}(\mathbf{C}) = U(N)/U(N-M)\times U(M)$
BDI	$Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2)\times U(1)$
CI	$Sp(N)/U(N)$
DIII	$SO(2N)/U(N)$
EIII	$E_6/SO(10)\times U(1)$
EVII	$E_7/E_6\times U(1)$

2D, $\mathcal{N} = 2$ (=dim.red.of 4D, $\mathcal{N} = 1$) Supersymmetric Nonlinear Sigma Models (SNLσM)

We use the notation of 4D, $\mathcal{N} = 1$.

Chiral superfields:

$$\Phi^i(y,\theta) = \varphi^i(y) + \theta\psi^i(y) + \theta\theta F^i(y)$$

$K(\Phi, \Phi^\dagger)$: Kähler potential

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta}\, K(\Phi, \Phi^\dagger) \\ &= g_{ij^*}(\varphi, \varphi^*)\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + \cdots\end{aligned}$$

$$g_{ij^*}(\varphi, \varphi^*) = \partial_i\partial_{j^*}K(\varphi, \varphi^*)$$

target = Kähler manifolds (Zumino)

The construction of SNLσM

1. Supersymmetric Nonlinear Realization (Bando-Kuramoto-Maskawa-Uehara, Phys. Lett. **138B** (1984) 94)
2. Auxiliary Field Formulation (Our results)

§2. Auxiliary Field Formulation of SNLσM

Result 1: We formulate all HSS G/H by the auxiliary field methods.

Result 2: We can perform the path integration over auxiliary fields ***exactly*** nevertheless they are *not* quadratic.

1. $\mathbf{C}P^{N-1} = SU(N)/SU(N-1) \times U(1)$

   (known)

   $SU(N) \times \underline{U(1)_{\mathrm{D}}}$ *gauging*

   $\vec{\phi} \in (\mathbf{N}, 1)$

   V: auxiliary vector superfield

$$\mathcal{L} = \int d^4\theta (e^V \phi^\dagger \phi - cV)$$

   cV : Fayet-Iliopoulos term

   gauge transformation:

   $e^{i\Lambda(x,\theta,\bar{\theta})} \in U(1)^{\mathbf{C}}$

   $\vec{\phi} \to e^{i\Lambda}\vec{\phi}, \quad e^V \to e^V e^{-i\Lambda + i\Lambda^\dagger}$

2. $G_{N,M}(\mathbf{C}) = U(N)/U(N-M) \times U(M)$
(known)

$U(N) \times \underline{U(M)}$ $(N > M)$
$\Phi \in (\mathbf{N}, \bar{\mathbf{M}})$: $N \times M$ matrix
$V = V^A T_A$: $M \times M$ matrix valued
auxiliary vector superfields
(T_A are Lie algebras of $U(M)$)

$$\mathcal{L} = \int d^4\theta \left(\mathrm{tr}\left(\Phi^\dagger \Phi e^V \right) - c\, \mathrm{tr} V \right)$$

3. $Q^{N-2}(\mathbf{C}) = SO(N)/SO(N-2) \times U(1)$

$SO(N) \times \underline{U(1)_{\mathrm{D}}}$
$\vec{\phi} \in (\mathbf{N}, 1)$,
$\phi_0 \in (\mathbf{1}, -2)$: auxiliary chiral superfield

$$\mathcal{L} = \int d^4\theta (e^V \vec{\phi}^\dagger \vec{\phi} - cV) + \left(\int d^2\theta\, \phi_0 \vec{\phi}^2 + \mathrm{c.c.} \right)$$

- Integration over V gives $\mathbf{C}P^{N-1}$.
- Integration over ϕ_0 gives the F-term constraint $\vec{\phi}^2 = 0$.

$\mathbf{C}P^{N-1}$ with an F-term constraint $\vec{\phi}^2 = 0$

4. $SO(2N)/U(N)$

$SO(2N) \times \underline{U(N)}$
$\Phi \in (\mathbf{2N}, \bar{\mathbf{N}}, 1)$,
$\Phi_0 \in (\mathbf{1}, \text{sym. tensor}, -2)$
$J^T = J$: rank-2 SO-invariant tensor

$$\mathcal{L} = \int d^4\theta \left(\mathrm{tr}\left(\Phi^\dagger \Phi e^V\right) - c\,\mathrm{tr}V\right) + \left(\int d^2\theta\ \mathrm{tr}\left(\Phi_0 \Phi^T J \Phi\right) + \text{c.c.}\right)$$

$G_{2N,N}$ with F-term constraints $\Phi^T J \Phi = 0$

5. $Sp(N)/U(N)$

$Sp(N) \times \underline{U(N)}$
$\Phi \in (\mathbf{2N}, \bar{\mathbf{N}}, 1)$,
$\Phi_0 \in (\mathbf{1}, \text{anti-sym. tensor}, -2)$
$J^T = -J$: rank-2 Sp-invariant tensor

$$\mathcal{L} = \int d^4\theta \left(\mathrm{tr}\left(\Phi^\dagger \Phi e^V\right) - c\,\mathrm{tr}V\right) + \left(\int d^2\theta\ \mathrm{tr}\left(\Phi_0 \Phi^T J \Phi\right) + \text{c.c.}\right)$$

$G_{2N,N}$ with F-term constraints $\Phi^T J \Phi = 0$

6. $E_6/SO(10) \times U(1)$

$E_6 \times \underline{U(1)_{\rm D}}$
$\vec{\phi} \in (\mathbf{27}, 1), \quad \vec{\phi}_0 \in (\mathbf{27}, -2)$
Γ_{ijk}: rank-3 E_6 inv. symmetric tensor

$$\mathcal{L} = \int d^4\theta (e^V \vec{\phi}^\dagger \vec{\phi} - cV) + \left(\int d^2\theta \, {\phi_0}^i \Gamma_{ijk} \phi^j \phi^k + \text{c.c.} \right)$$

$\mathbf{C}P^{N-1}$ with constraints $\Gamma_{ijk}\phi^j\phi^k = 0$

7. $E_7/E_6 \times U(1)$

$E_7 \times \underline{U(1)_{\rm D}}$
$\vec{\phi} \in (\mathbf{56}, 1), \quad \vec{\phi}_0 \in (\mathbf{56}, -3)$
$d_{\alpha\beta\gamma\delta}$: rank-4 E_7 inv. symmetric tensor

$$\mathcal{L} = \int d^4\theta (e^V \vec{\phi}^\dagger \vec{\phi} - cV) + \left(\int d^2\theta \, {\phi_0}^\alpha d_{\alpha\beta\gamma\delta} \phi^\beta \phi^\gamma \phi^\delta + \text{c.c.} \right)$$

$\mathbf{C}P^{N-1}$ with constraints $d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0$

§3. Path Integration over Auxiliary Fields

Quantum Legendre transformation

$\sigma(x,\theta,\bar{\theta})$, $\Phi(x,\theta,\bar{\theta})$: vector superfields

$$\int[d\sigma]\exp\left[i\int d^4xd^4\theta\,(\sigma\Phi - W(\sigma))\right]$$
$$= \exp\left[i\int d^4xd^4\theta\, U(\Phi)\right]$$

$U(\Phi) = \hat{\sigma}(\Phi)\Phi - W(\hat{\sigma}(\Phi))$ (Legendre tr. of W)

$\hat{\sigma}$ is the stationary path:

$$\frac{\partial}{\partial\sigma}(\sigma\Phi - W(\sigma))|_{\sigma=\hat{\sigma}} = \Phi - \partial W(\hat{\sigma}) = 0$$

The path integration can be performed exactly!!

Integration over (gauge)vector superfields

$\sigma = e^V, \quad \Phi = \phi^\dagger\phi, \quad W(\sigma) = c\log\sigma = cV$

$\sigma = f(V) \Rightarrow$ We can show $[d\sigma] = [dV]$

$$\int[dV]\exp\left[i\int d^4xd^4\theta\left(e^V\phi^\dagger\phi - cV\right)\right]$$
$$= \exp\left[i\int d^4xd^4\theta\, c\log(\phi^\dagger\phi)\right]$$

Path integration over V
= Elimination by using eq. of motion

Proof

$$
\begin{aligned}
&\int d^4\theta\,(\sigma\Phi - W(\sigma)) \\
&= \frac{1}{2}D_\sigma(C_\Phi - W'(C_\sigma)) \\
&\quad -\frac{1}{2}\lambda_\sigma(\chi_\Phi - W''(C_\sigma)\chi_\sigma) - \frac{1}{2}\bar{\lambda}_\sigma(\bar{\chi}_\Phi - W''(C_\sigma)\bar{\chi}_\sigma) \\
&\quad +\frac{1}{2}C_\sigma D_\Phi - \frac{1}{2}(\chi_\sigma\lambda_\Phi + \bar{\chi}_\sigma\bar{\lambda}_\Phi) \\
&\quad +\frac{1}{2}(M_\sigma M_\Phi + N_\sigma N_\Phi - v_\sigma \cdot v_\Phi) \\
&\quad -\frac{1}{4}W''(C_\sigma)({M_\sigma}^2 + {N_\sigma}^2 - {v_\sigma}^2) \\
&\quad +\frac{1}{8}W'''(C_\sigma)[i(\chi_\sigma\chi_\sigma - \bar{\chi}_\sigma\bar{\chi}_\sigma)M_\sigma + (\chi_\sigma\chi_\sigma + \bar{\chi}_\sigma\bar{\chi}_\sigma)N_\sigma \\
&\qquad\qquad +2v_{\sigma\mu}(\chi_\sigma\sigma^\mu\bar{\chi}_\sigma)] \\
&\quad -\frac{1}{16}W''''(C_\sigma)\chi_\sigma\chi_\sigma\bar{\chi}_\sigma\bar{\chi}_\sigma
\end{aligned}
$$

The path integration coincides with the substitution of the solution of stationary equations. (Q.E.D.)

Vector superfields:

$$
\begin{aligned}
&V(x,\theta,\bar{\theta}) \\
&= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
&\quad +\frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\
&\quad -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) \\
&\quad +\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)
\end{aligned}
$$

§4. Conclusion

- We formulated 2D, $\mathcal{N} = 2$ SNLσM on Hermitian Symmetric Spaces (HSS) by the auxiliary field methods.
- We needed two kinds of auxiliary superfields:
  1. auxiliary chiral superfields
     $\Rightarrow$ F-term constraints
  2. auxiliary vector superfields
     $\Rightarrow$ D-term constraints
- We performed the path integration over auxiliary superfields *exactly* nevertheless they are not quadratic.

Application

Non-perturbative analysis (large-N) of 2D, $\mathcal{N} = 2$ SNLσM on HSS.

Work in progress in collaboration with K. Higashijima, T. Kimura and M. Tsuzuki

G/H	$G^{\mathbf{C}}$-invariants	superpotentials	constraints	embedding
$\frac{SO(N)}{SO(N-2)\times U(1)}$	$I_2 = \vec{\phi}^T J \vec{\phi}$	$\phi_0 I_2$	$I_2 = 0$	$\mathbf{C}P^{N-1}$
$\frac{SO(2N)}{U(N)}, \frac{Sp(N)}{U(N)}$	$I_2' = \Phi^T J \Phi$	$\mathrm{tr}(\Phi_0 I_2')$	$I_2' = 0$	$G_{2N,N}$
$\frac{E_6}{SO(10)\times U(1)}$	$I_3 = \Gamma_{ijk}\phi^i\phi^j\phi^k$	$\Gamma_{ijk}{\phi_0}^i\phi^j\phi^k$	$\partial I_3 = 0$	$\mathbf{C}P^{26}$
$\frac{E_7}{E_6\times U(1)}$	$I_4 = d_{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta$	$d_{\alpha\beta\gamma\delta}{\phi_0}^\alpha\phi^\beta\phi^\gamma\phi^\delta$	$\partial I_4 = 0$	$\mathbf{C}P^{55}$