INSTANTON CALCULUS AND TOPOLOGICAL FIELD THEORIES

- The non perturbative sector of a field theory

- Recent progresses: Seiberg–Witten, AdS/CFT, D–branes

- Conjectures need checks: multi–instanton Calculus

- The special role of N=2: the power of Holomorphicity

- First checks give hints of "topology"

- Rivisiting the computation:

i) New computational advantages

ii) Can we say anything on old problems?
an old controversy: scalar v.e.v. =0 versus v.e.v. Different from zero

- Measure for SUSY theory

A BRIEF REVIEW OF TFT

- The symmetry group

$$SU(2)_L \times SU(2)_R \times SU(2)_A \times U(1)_R$$

Taking its diagonal subgroup leads to

$$\bar{Q}_{\dot{\alpha}}^{\dot{A}} \to Q \oplus Q_{\mu\nu} \ ,$$

$$Q_{\alpha}^{\dot{A}} \to Q_{\mu} \ .$$

And

$$A_\mu \to A_\mu \ ,$$

$$\bar{\lambda}_{\dot{\alpha}}^{\dot{A}} \to \eta \oplus \chi_{\mu\nu} \ ,$$

$$\lambda_{\alpha}^{\dot{A}} \to \psi_\mu \ ,$$

$$\phi \to \phi \ .$$

THE ADHM CONSTRUCTION

- An algebraic problem replaces a differential one

$$\Delta = a + bx \ ,$$

$$\Delta^\dagger \Delta = (\Delta^\dagger \Delta)^T$$

$$a = \begin{pmatrix} w_1 & \dots & w_k \\ & & \\ & a' & \\ & & \end{pmatrix} \ ;$$

$$A = U^\dagger dU$$

- Symmetries of the construction

$$\Delta \to Q \Delta R$$

Where

$$Q \in Sp(k+1), R \in GL(k, \mathbb{R})$$

- Plugging the zero–modes into the BRST variations

$$
\begin{aligned}
s\Delta &= \mathcal{M} - \mathcal{C}\Delta \ , \\
s\mathcal{M} &= \mathcal{A}\Delta - \mathcal{C}\mathcal{M} \ , \\
s\mathcal{A} &= -[\mathcal{C}, \mathcal{A}] \ , \\
s\mathcal{C} &= \mathcal{A} - \mathcal{C}\mathcal{C} \ ,
\end{aligned}
$$

Where

$$
\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}' & \\ 0 & & & \end{pmatrix}
$$

And s squared is zero

- How does a computation looks like?

$$\langle fields \rangle = \int_{\mathcal{M}^+} \left[(fields)\ e^{-S_{\rm TYM}} \right]_{zero-mode\ \ subspace}$$

For a definite value of the winding k

$$\begin{aligned}
[S_{\rm inst}]_k &= 4\pi^2 s{\rm Tr}\left[\bar{\mathcal{A}}_{00} \sum_{l=1}^{k} (\mu_l \bar{w}_l - w_l \bar{\mu}_l)\right] \\
&= 4\pi^2 s{\rm Tr}\left[\bar{\mathcal{A}}_{00} (\mathcal{M}\Delta^\dagger - \Delta\mathcal{M}^\dagger)_{00}\right] = -4\pi^2 s{\rm Tr}\left[\bar{\mathcal{A}}_{00}\left(\Delta \overleftrightarrow{s} \Delta^\dagger\right)_{00}\right]
\end{aligned}$$

- For a function of Grassmann variables

$$\begin{aligned}
g(\widehat{\Delta}, \widehat{\mathcal{M}}) &= g_0(\widehat{\Delta}) + g_{i_1}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1} + \frac{1}{2!} g_{i_1 i_2}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2} + \cdots \\
&+ \frac{1}{p!} g_{i_1 i_2 \ldots i_p}(\widehat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2} \cdots \widehat{\mathcal{M}}_{i_p} \ ,
\end{aligned}$$

Now from the BRST variations

$$\widehat{\mathcal{M}}_i = K_{ij}(\widehat{\Delta}) s \widehat{\Delta}_j$$

Leading to

$$\int_{\mathcal{M}^+} g(\hat{\Delta}, \widehat{\mathcal{M}}) = \frac{1}{p!}\int_{\mathcal{M}^+} g_{i_1 i_2 \dots i_p}(\hat{\Delta})\widehat{\mathcal{M}}_{i_1}\widehat{\mathcal{M}}_{i_2}\cdots\widehat{\mathcal{M}}_{i_p} =$$
$$= \int_{\mathcal{M}^+} s^p\hat{\Delta}\,|\det K| g_{12\dots p}(\hat{\Delta}) \ .$$

- The use of the derivative s:

$$[S_{\text{inst}}]_k = [S_B + S_F]_k$$

Then

$$e^{-[S_{\text{inst}}]_k}\Big|_{top\ form} =$$

$$4\pi^2 s\left\{\mathrm{Tr}\Big[\bar{v}(\sum_{i=1}^{k}\mu_i\bar{w}_i - w_i\bar{\mu}_i)\Big]([S_F]_k)^{4k-3}([S_B]_k)^{-4k+2}\right.$$
$$\left.\Big(1 - e^{-[S_B]_k}\sum_{l=0}^{4k-3}\frac{([S_B]_k)^l}{l!}\Big)\right\} \ .$$

→ Correlators are functions at $\partial\overline{\mathcal{M}}_k$

- How does the boundary look like?

$$\overline{\mathcal{M}}_k = \mathcal{M}_k \cup \mathbb{R}^4 \times \mathcal{M}_{k-1} \cup S^2\mathbb{R}^4 \times \mathcal{M}_{k-2} \ldots \cup S^k\mathbb{R}^4$$

For something familiar

$$|F_k|^2 = |F_{k-l}|^2 + \sum_{i=1}^{l} 8\pi^2 \delta(x - y_i)$$

- An example: k=2

$$\Delta = \begin{pmatrix} w_1 & w_2 \\ x_1 - x & a_1 \\ a_1 & x_2 - x \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix} + b(x - x_0)$$

$$\lim_{|w_1| \to 0} \Delta \longrightarrow \begin{pmatrix} 0_{2\times 1} & \Delta_{k=1} \\ x_1 - x & 0_{1\times 1} \end{pmatrix}$$

$$\lim_{|w_1|\to 0} \mathrm{Tr}(FF)_{k=2} = -\frac{1}{2} \lim_{|w_1|\to 0} \Box\Box \log\det(\Delta^\dagger \Delta)_{k=2} d^4x$$

$$= -\frac{1}{2}\Box\Box \log(x_1 - x)^2 d^4x + \mathrm{Tr}(FF)_{k=1} = \mathrm{Tr}(FF)_{k=1} + 8\pi^2\delta^4(x - x_1)$$

CONCLUSIONS

- Some old questions in a different perspective

  i) weak vs. strong coupling
  ii) constrained instanton
  iii) clustering

- Useful tools for a recursion relation?

$$u = \Sigma_k G_k \left(\frac{\Lambda}{a}\right)^{4k} a^2 ; G_2 = \frac{1}{32 G_0^2}(3G_1 - 4G_0 G_1^2)$$

$$F_2 = \frac{1}{32}(F_1 + 4aF_1' + 2a^2 F_1'')$$

- Topology, integrable systems 4d vs. 2d

THE HYPERKHALER QUOTIENT COSTRUCTION

- The SU(N) case: out of the data

$$a = \begin{pmatrix} t & s^\dagger \\ A & -B^\dagger \\ B & A^\dagger \end{pmatrix}$$

Build a $4k^2+4kN$ Manifold $M = \{A, B, s, t\}$

Define a 2–form $\omega^i = J^i_{ab} dx^a \wedge dx^b$

$$\begin{aligned} \omega_{\mathbb{C}} &= \mathrm{Tr}\, dA \wedge dB + \mathrm{Tr}\, ds \wedge dt \ , \\ \omega_{\mathbb{R}} &= \mathrm{Tr}\, dA \wedge dA^\dagger + \mathrm{Tr}\, dB \wedge dB^\dagger + \mathrm{Tr}\, ds \wedge ds^\dagger - \mathrm{Tr}\, dt^\dagger \wedge dt \ . \end{aligned}$$

$$\begin{aligned} A &\rightarrow QAQ^\dagger \ , \\ B &\rightarrow QBQ^\dagger \ , \\ s &\rightarrow QsR^\dagger \ , \\ t &\rightarrow RtQ^\dagger \ , \end{aligned}$$

- The isometries of the 2–form

$$L_\xi \omega^i = 0$$

Generate conserved quantities

$$i(\xi)\omega^i = d\mu^i_\xi$$

$$\begin{aligned}\mu_{\mathbb{C}} &= [A,B] + st \ , \\ \mu_{\mathbb{R}} &= [A,A^\dagger] + [B,B^\dagger] + ss^\dagger - t^\dagger t\end{aligned}$$

A new variety

$$\mathcal{N}^+ = \left\{ \{A,B,s,t\} = x \in M : \ \ \mu^i_\xi = 0 \right\}$$

To be quotiented

- Example k=2

$$ds^2 = \eta_{I\bar{J}} dm^I d\bar{m}^{\bar{J}} = |dw_1|^2 + |dw_2|^2 + |da_3|^2 + |da_1|^2$$

The constraint

$$\bar{w}_2 w_1 - \bar{w}_1 w_2 = 2(\bar{a}_3 a_1 - \bar{a}_1 a_3)$$

$$(w_1^\theta, w_2^\theta) = (w_1, w_2) R_\theta \quad ,$$

$$(a_3^\theta, a_1^\theta) = (a_3, a_1) R_{2\theta} \quad ,$$

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad .$$

This invariance is generated by

$$\mathcal{C} = \frac{1}{|k|^2} \eta_{I\bar{J}} (\bar{k}^{\bar{J}} dm^I + d\bar{m}^{\bar{J}} k^I)$$

$$\begin{aligned} \mathcal{C} &= \frac{1}{H}\Big(\bar{w}_1 dw_2 - \bar{w}_2 dw_1 + 2\bar{a}_3 da_1 - 2\bar{a}_1 da_3 + \\ &\quad + d\bar{w}_2 w_1 - d\bar{w}_1 w_2 + 2d\bar{a}_1 a_3 - 2d\bar{a}_3 a_1\Big) \quad . \end{aligned}$$

- The final metric

$$L = g^{\mathcal{M}^+}_{AB} sm^A sm^B = \left(g^{\mathcal{N}+}_{AB} - \frac{g^{\mathcal{N}+}_{AC} g^{\mathcal{N}+}_{BD} k^C k^D}{g^{\mathcal{N}+}_{EF} k^E k^F}\right) sm^A sm^B$$

$$= g^{\mathcal{N}+}_{AB} sm^A sm^B - C^{12} C_{12} = g^{\mathcal{N}+}_{AB} \mathcal{S}m^A \mathcal{S}m^B$$

And measure

$$|w_1|^4 \sqrt{g^{\mathcal{M}+}_{\Sigma=0}} d^4 w_1 d^4 w_2 d^4 a_3 d^4 x_0 =$$

$$\frac{H}{|a_3|^4} \left| |a_3|^2 - |a_1|^2 \right| d^4 w_1 d^4 w_2 d^4 a_3 d^4 x_0$$

- A Kaehler potential

$$\mathcal{K} = \int_{\mathbb{R}^4} x^2 |F|^2 = \frac{1}{2} \mathrm{Tr}\left[a^\dagger (1 + P_\infty) a\right]$$

$$\omega_{\mathcal{M}^+} = \partial \bar{\partial} \mathcal{K} = \frac{1}{2} \mathrm{Tr}\left[(\mathcal{S}a)^\dagger (1 + P_\infty) \mathcal{S}a\right]$$

From which

$$L' = \mathcal{S}m^I \mathcal{S}m^I = (\mathcal{S}a)^\dagger (1 + P_\infty)\mathcal{S}a = sm^I sm^I - CCH$$

- The action $S_{TYM} = S_B + S_F$

$$[S_B]_k = 4\pi^2 \mathrm{Tr}\Big[2(\bar{v}v)\sum_{l=1}^{k}|w_l|^2 + \sum_{l,p=1}^{k}(\bar{w}_l\bar{v}w_p - \bar{w}_p\bar{v}w_l)(\mathcal{A}'_b)_{lp}\Big]$$

$$[S_F]_k = 4\pi^2 \mathrm{Tr}\Big[-2\bar{v}\sum_{l=1}^{k}\mu_l\bar{\mu}_l + \sum_{l,p=1}^{k}(\bar{w}_l\bar{v}w_p - \bar{w}_p\bar{v}w_l)(\mathcal{A}'_f)_{lp}\Big]$$

Where

$$[S_F]_k = \overline{\mathcal{M}}_i^{\dot{A}\alpha}(h_{ij})_\alpha{}^\beta(\mathcal{M}_j)_{\beta\dot{A}} = (\mathcal{S}_i^{\dot{A}\alpha}a)^\dagger(h_{ij})_\alpha{}^\beta(\mathcal{S}_j)_{\beta\dot{A}}a$$

- A prescription

$$(fields) = \int_{\mathcal{M}^+} dC d\mathcal{A}\delta(C + L^{-1}\Lambda_C)\delta(\Delta^\dagger \mathcal{A}\Delta - (\Delta^\dagger \mathcal{A}\Delta)^T)\delta((\Delta^\dagger\mathcal{M}) - (\Delta^\dagger\mathcal{M})^T)$$
$$\delta(\mathcal{M} - \mathcal{S}\Delta)\delta(\frac{1}{4}tr_2\sigma^a(\Delta^\dagger\Delta - (\Delta^\dagger\Delta)^T)\delta(f(C))\left[(fields)\ e^{-S_{TYM}}\right]_{zero-mode},$$