Weak matrix elements and K-meson physics

Massimo Testa
University of Rome "La Sapienza"

Kaon Physics still defies complete physical understanding: it is a very complicated blend of Ultraviolet and Infrared effects

The problem is:

$$\frac{\Gamma(K^+ \to \pi^+\pi^0)}{\Gamma(K_s^0 \to \pi^+\pi^-)} \approx 1/400 \approx \frac{|\Delta I = 3/2|^2}{|\Delta I = 1/2, 3/2|^2}$$

$$\Rightarrow A(\Delta I = 1/2) \approx 20\, A(\Delta I = 3/2)$$

Being a process involving hadrons, it must be treated non-perturbatively.

Due to the difficulty of putting the Standard Model on the lattice (for **technical** ($M_{W,Z} >> 1/a$) & **theoretical** reasons (breaking of chiral gauge invariance in regularisation)) we must "integrate" analytically the heavy degrees of freedom (W's, Z's)

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- Effective low energy actions for non-leptonic decays:

$$H_{eff} = g_W^2 \int d^4x D(x; M_W) T[J_L^\mu(x) J_{L\mu}^\dagger(0)] + c_m O_m$$

where:

$$D(x;M_W) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{\exp(ipx)}{(p^2 + M_W^2)}$$

$$O_m \equiv (m_s + m_d)\bar{s}d + (m_s - m_d)\bar{s}\gamma_5 d + h.c.$$

and c_m is fixed, in Perturbation Theory, so that quark masses are not modified by Weak Interactions.

The term proportional to O_m does not contribute to the physical amplitudes, being the four-divergence of the Axial current ($\Delta p_\mu = 0$).

Since $M_W >> \Lambda_{QCD}$ we can use Operator Product Expansion:

$$H^{eff}_{\Delta S=1} = \lambda_u \frac{G_F}{\sqrt{2}} \left[C_+(\mu) O^{(+)}(\mu) + C_-(\mu) O^{(-)}(\mu) \right]$$

where $\lambda_u = V_{ud} V^*_{us}$ and:

$$O^{(\pm)} = \left[(\bar{s}\gamma^L_\mu d)(\bar{u}\gamma^L_\mu u) \pm (\bar{s}\gamma^L_\mu u)(\bar{u}\gamma^L_\mu d) \right] - \left[u \leftrightarrow c \right]$$

$O^{(-)}$ is a pure $I = \frac{1}{2}$, while $O^{(+)}$ is a mixture of $I = \frac{1}{2}$ and $I = \frac{3}{2}$.

The $O^{(\pm)}$'s have definite transformation properties under the $SU(3)\otimes SU(3)$ chiral group (and some discrete symmetries): they transform as $(8,1)$ and $(27,1)$

The $C_{\pm}(\mu)$, can be reliably computed in Perturbation Theory, due to Asymptotic Freedom, and show a slight octet enhancement:

$$\left|\frac{C_{-}(\mu \approx 2\,Gev)}{C_{+}(\mu \approx 2\,Gev)}\right| \approx 2$$

The rest of the enhancement (≈ 10) should be provided by the matrix elements of $O^{(\pm)}$ and is a non perturbative, infrared effect.

As for Lattice discretisation (regularisation) the difficulty lies in the fact that naive discretisation of Dirac fermions entails a multiplication of low energy degrees of freedom (Doublers) whose elimination complicates the scheme.

There are, essentially, two possibilities:

a) Wilson Fermions

A term is added to the Lagrangian, breaking explicitly the chiral symmetry, which can be restored, as $a \to 0$, by the inclusion of appropriate counterterms.

This formulation is ultra-local (at the lagrangian level only near neighbours interactions are involved) and therefore it is very convenient for numerical purposes.

b) Ginsparg-Wilson Fermions

(Narayanan-Neuberger 1993, Lüscher, Hasenfratz 1998)

This discretisation is much more respectful of the chiral properties of the (continuum) QCD lagrangian, at the expense of being non local at the lattice level, which makes it, at the moment, numerically very demanding.

In this talk, therefore, my remarks on renormalisation, will be addressed to Wilson fermions

The difficulty of the problem consists, first of all, in giving the correct definition of the operators $\boxed{O^{(\pm)}}$

In order to construct finite composite operator of dimension 6, $\tilde{O}_6(\mu)$, we must

mix the original bare operator, $\boxed{O_6(a)}$, with bare operators of equal ($\boxed{O_6^{(i)}(a)}$) or smaller ($\boxed{O_3(a)}$) dimension, in general with different naive chiralities

We have, schematically:

$$\tilde{O}_6(\mu) = Z(\mu a)\left[O_6(a) + \sum_i C_i(g_0) O_6^{(i)}(a) + \frac{C_3(g_0)}{a^3} O_3(a)\right]$$

Being QCD asymptotically free, one could think that the mixing coefficients could be reliably computed in Perturbation Theory.

Numerical attempts (and theoretical considerations) show that this is not necessarily the case

We should therefore employ a general non perturbative technique

Such a technique is based on the systematic exploitation of (continuum) symmetries.

This suggests the use of Chiral Ward Identities (or equivalent methods) to classify the composite operators.

(L.Maiani, G.Martinelli, G.C.Rossi, M.Testa 1987)

Infrared Problems (common to all approaches)

(Continuum)

1) **Infinite volume**

In order to compute the $K \to \pi\pi$ width we have to evaluate the matrix element:

$$ {}_{(\text{out})}\langle \pi(\underline{p})\pi(-\underline{p})|H_W|K\rangle $$

with two interacting hadrons in the final state.

This is not easy to do in the euclidean region

It can be shown (L.Maiani, M.Testa 1990) that:

$$ \langle \varphi_{\underline{p}}(t_1)\varphi_{-\underline{p}}(t_2)H_W(0)K(t_K)\rangle \underset{\substack{t_K \to -\infty \\ t_1 >> t_2 >> 0}}{\approx} $$

$$ \approx e^{m_K t_K - E_{\underline{p}} t_1 - E_{\underline{p}} t_2} \sqrt{\frac{Z_K}{2m_K}\frac{Z_\pi}{2m_\pi}} \left\{ \left[\frac{{}_{(out)}\langle \underline{p},-\underline{p}|H_W|K\rangle + {}_{(in)}\langle \underline{p},-\underline{p}|H_W|K\rangle}{2} \right] + \right. $$

$$ \left. + P_{\underline{q}}(t_2) \right\} $$

$$P_{\underline{q}}(t_2) = -P\sum_n \exp[-(E_n - 2E_{\underline{q}})t_2](2\pi)^3\delta^3(\underline{P}_n)$$

$$N_n \frac{[M(\underline{q},-\underline{q};n)]^* \langle n, out|H_W(0)|K\rangle}{E_n(E_n - 2E_{\underline{q}})}$$

$$P_{\underline{q}}(t_2) \approx e^{2(E_{\underline{q}} - m_\pi)t_2}$$

where $\mathrm{M}^{(\mathrm{o.s.})}_{\underset{2m_\pi}{\pi\pi} \to \underset{2E_\pi}{\pi\pi}}$ is an off-shell extrapolation of the $(\underset{2m_\pi}{\pi\pi} \to \underset{2E_\pi}{\pi\pi})$ scattering amplitude (final state interaction).

If $\underline{p} = 0$ we have:

$$\left\langle \pi_{\underline{0}}(t_1)\pi_{\underline{0}}(t_2)H_W(0)K(t_K)\right\rangle \underset{\substack{t_K \to -\infty \\ t_1 >> t_2 >> 0}}{\approx}$$

$$\approx e^{m_K t_K - m_\pi t_1 - m_\pi t_2}\langle \pi(\underline{0})\pi(\underline{0})|H_W|K\rangle(1 + \frac{c}{\sqrt{t_2}}\mathrm{M}^{(\mathrm{o.s.})}_{\underset{2m_\pi}{\pi\pi} \to \underset{2m_\pi}{\pi\pi}})$$

Only for π's at rest (and weakly interacting) it is possible to extract a meaningful matrix element

A possible strategy is, therefore, to compute $K \to \pi\pi$ with the two pions at rest, choosing the quark masses such that $m_K = 2m_\pi$ or $m_K = m_\pi$ and then extrapolating to the real world through chiral perturbation theory.

These choices do not require any renormalisation, apart from the overall normalisation

2) Finite volume

Lellouch and Lüscher (2000) have proposed a variant of this (infrared) strategy, based on the exploitation of the finiteness of volume in lattice simulations.

They find a relation between finite and infinite volume matrix elements:

$$\left|\langle \pi\pi, E = m_K \left| H_W(0) \right| K\rangle\right|^2 =$$

$$= V^2 \left|{}_V\langle \pi\pi, n \left| H_W(0) \right| K\rangle_V\right|^2 \left(\frac{m_K}{k_\pi}\right)^3 8\pi[\, q\phi'(q) + k\delta_0'(k)]$$

A simple demonstration

(D.Lin, G.Martinelli, C.Sachrajda, M.Testa in preparation)

In a finite volume the allowed values, k of the "radial" relative momentum of a two particle state obeys the relation:

$$n\pi - \delta_0(k) = \phi(q)$$

where $\delta_0(k)$ is the s-wave phase-shift, $q \equiv \frac{kL}{2\pi}$, k is related to the center of mass energy, E as:

$$E = 2\sqrt{m_\pi^2 + k^2}$$

and:

$$\tan\phi(q) = -\frac{\pi^{3/2} q}{Z_{00}(1;q^2)}$$

$$Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\underline{n} \in Z^3} (\underline{n}^2 - q^2)^{-s}$$

In order to relate the states at finite and infinite volume we consider the two-point Green function of a scalar operator $\sigma(x)$:

$$\int_V d^3x \langle \sigma(\underline{x},t)\sigma(0)\rangle = V\sum_n \left|\langle 0|\sigma(0)|\pi\pi,n\rangle_V\right|^2 e^{-E_n t} \xrightarrow[V\to\infty]{}$$

$$\xrightarrow[V\to\infty]{} V\int_0^\infty dE\rho(E)\left|\langle 0|\sigma(0)|\pi\pi,E\rangle_V\right|^2 e^{-Et}$$

where $|\pi\pi,n\rangle_V$, $|\pi\pi,E\rangle_V$ and $|K\rangle_V$ denote finite volume states with zero total momentum (normalised to 1) and:

$$\rho(E) \equiv \frac{\Delta n}{\Delta E} = \frac{q\phi'(q) + k\delta_0'(k)}{4\pi k^2} E$$

denotes the density of states of given energy E.

On the other hand:

$$\int_V d^3x \langle \sigma(\underline{x},t)\sigma(0)\rangle_V \xrightarrow[V\to\infty]{}$$

$$\xrightarrow[V\to\infty]{} \frac{(2\pi)^3}{2(2\pi)^6}\int \frac{d\underline{p}_1}{2\omega_1}\frac{d\underline{p}_2}{2\omega_2}\delta(\underline{p}_1+\underline{p}_2)e^{-(\omega_1+\omega_2)t}\left|\langle 0|\sigma(0)|\underline{p}_1,\underline{p}_2\rangle\right|^2 =$$

$$= \frac{1}{2(2\pi)^3}\int dE e^{-Et}\left|\langle 0|\sigma(0)|\pi\pi,E\rangle\right|^2 \int \frac{d\underline{p}_1}{2\omega_1}\frac{d\underline{p}_2}{2\omega_2}\delta(\underline{p}_1+\underline{p}_2)\delta(E-\omega_1-\omega_2) =$$

$$= \frac{\pi}{2(2\pi)^3}\int \frac{dE}{E} e^{-Et}\left|\langle 0|\sigma(0)|E\rangle\right|^2 k(E)$$

where

$$k(E)=\sqrt{\frac{E^2}{4}-m_\pi^2}$$

and $|\pi\pi,E\rangle$ and $|K\rangle$ now denote infinite volume states of zero total momentum (of total energy E), normalised as:

$$\langle \underline{p}|\underline{q}\rangle=(2\pi)^3 2\omega_{\underline{q}}\delta^{(3)}(\underline{p}-\underline{q})$$

By comparison we get:

$$|\pi\pi,E\rangle \Leftrightarrow 4\pi\sqrt{\frac{VE\rho(E)}{k(E)}}|\pi\pi,E\rangle_V$$

Similarly:

$$|\underline{p}=0\rangle \Leftrightarrow \sqrt{2mV}|\underline{p}=0\rangle_V$$

so that:

$$|\langle\pi\pi,E=m_K|H_W(0)|K\rangle|^2=$$
$$=V^2|_V\langle\pi\pi,n|H_W(0)|K\rangle_V|^2\left(\frac{m_K}{k_\pi}\right)^3 8\pi[q\phi'(q)+k\delta_0'(k)]$$

The strategy is as follows

Tune the space volume V so that the first excited two-pion state ($n=1$) is degenerate in energy with the kaon state ($L \approx 5 \div 6\, Fm$) and consider the finite volume Green's functions:

$$\int_V d^3x d^3y \langle \sigma(\underline{x},t) H_W(0) K(\underline{y},t') \rangle_V \underset{t' \to -\infty}{\approx}$$

$$= e^{m_K t'}\, {}_V\langle K | K(0) | 0 \rangle V^2 \sum_n \langle 0 | \sigma(0) | \pi\pi, n \rangle_V\, {}_V\langle \pi\pi, n | H_W(0) | K \rangle_V e^{-E_n t} =$$

$$= e^{m_K t'}\, {}_V\langle K | K(0) | 0 \rangle V^2 \sum_n \left| \langle 0 | \sigma(0) | \pi\pi, n \rangle_V \right| \left| {}_V\langle \pi\pi, n | H_W(0) | K \rangle_V \right| e^{-E_n t}$$

because final state interactions phases cancel.

Then, from:

$$\int_V d^3x \langle \sigma(\underline{x},t) \sigma(0) \rangle = V \sum_n \left| \langle 0 | \sigma(0) | \pi\pi, n \rangle_V \right|^2 e^{-E_n t}$$

we compute $\left| \langle 0 | \sigma(0) | \pi\pi, 1 \rangle_V \right|$ and, finally, $\left| {}_V\langle \pi\pi, 1 | H_W(0) | K \rangle_V \right|$.

Remarks

a) The relation between finite and infinite volume decay rate becomes valid when the sum over energy states can be approximated (up to exponentially small terms in the volume) by the corresponding integral.

This requires:

$$\left|{}_V\langle \pi\pi, 1|H_W(0)|K\rangle_V\right| \approx \left|{}_V\langle \pi\pi, 0|H_W(0)|K\rangle_V\right|$$

which hints at a connection with the methods which put the pions at rest.

b) In the case of a $\Delta I = \frac{1}{2}$ transition, within any procedure, we must subtract a disconnected contribution:

$$\langle \sigma(t) H_W(0) K(t')\rangle_{disc} = \langle \sigma(0)\rangle\langle H_W(0) K(t')\rangle$$

Operator Product Expansion

(C.Dawson, G.Martinelli, G.C.Rossi, C.T.Sachrajda, S.Sharpe, M.Talevi, M.Testa 1997)

The distortion of the composite operators in the effective Weak Hamiltonian is due to the fact that, in the computation, we insist to integrate up to zero (lattice) distance.

If we could integrate up to a finite (physical) distance and perform analytically the final, continuum, integration the problem would be overcome.

In order to accomplish this, we start again from the definition of the Weak Hamiltonian:

$$\langle h|H_{eff}|h'\rangle =$$
$$= g_W^2 \int d^4x D(x;M_W)\langle h|T[J_L^\mu(x)J_{L\mu}^+(0)]|h'\rangle + c\langle h|O_m|h'\rangle$$

$$\langle h|H_W^{eff}|h'\rangle = \frac{G_F}{\sqrt{2}}\sum_i C_i(\frac{\mu}{M_W})M_W^{6-d_i}\langle h|O^{(i)}(\mu)|h'\rangle$$

$$C_i(\frac{\mu}{M_W})M_W^{6-d_i} = \int dx D(x;M_W)c_i(x;\mu)$$

The $\boxed{C_i(\frac{\mu}{M_W})}$'s, and therefore the $\boxed{c_i(x;\mu)}$'s, are reliably computable within perturbation theory

It is then possible to get information directly on the continuum matrix elements $\langle h|O^{(i)}(\mu)|h'\rangle$ (including their normalization) by measuring:

$$\frac{1}{2}\left[\langle h|T[J_L^{\mu}(x)J_{L,\mu}^{+}(0)]|h'\rangle + x \leftrightarrow -x\right] = \sum_i c_i(x;\mu)\langle h|O^{(i)}(\mu)|h'\rangle$$

in the region:

$$a << |x| << \Lambda_{\mathrm{QCD}}^{-1}$$

If the weak currents are $O(a)$ improved, then the whole computation is automatically $O(a)$ improved.

In the case of $K \to \pi\pi$ we have the contribution of $O^{(\pm)}$ and $(m_c^2 - m_u^2)O_p$, in the parity violating case, or $(m_c^2 - m_u^2)(m_s + m_d)O_s$, in the parity conserving case.

$$c_i(x;\mu) \propto \left(\frac{\alpha_s(1/x)}{\alpha_s(\mu)}\right)^{\frac{\gamma_0^{(i)}}{2\beta_0}} \approx 1 + \frac{\alpha_s(\mu)}{4\pi}\gamma_0^{(i)}\log(x\mu) + \dots$$

$$\gamma^{(+)} = 4 \quad \gamma^{(-)} = -8 \quad \gamma' = 16$$

As before no chiral expansion is used and therefore this method could be applied also to B-decays

This method has been successfully tested by Caracciolo, Montanari and Pelissetto (1998-2000) in two dimensional σ-models

CONCLUSIONS AND OUTLOOK

Lattice discretisation is the only convergent (as $a \to 0$) approximation scheme to QCD.

Problems related to hadron physics (like $K \to \pi\pi$ and ε'/ε) can be reliably addressed and hopefully solved soon.