Exact chiral symmetry / Ginsparg-Wilson relation and Domain wall fermion

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In this talk

Chiral properties of low energy effective action of domain wall fermion

Its relation to exact chiral symmetry based on the Ginsparg-Wilson relation

- Vector-like theories
  Noguchi-Y.K. `hep-lat/9902022`
- Chiral gauge theories
  Aoyama-Y.K. `hep-lat/9905003`

Plan of Talk

1. Brief review on
   - Chiral symmetry breaking on the lattice
   - The Ginsparg-Wilson relation
   - Neuberger's lattice Dirac operator
2. Domain wall fermion and its low energy effective action
3. Chiral gauge theories using domain wall fermion

Recent Progress in Lattice Gauge Theory

- Exact Chiral Symmetry
- Gauge Interactions of Weyl Fermions

The Clue:

Gauge-covariant and local Lattice Dirac Operator
which satisfies the Ginsparg-Wilson relation

- Renormalization Group Approach
  (Fixed Point Action)

  P. Hasenfratz et al., NPB(Proc. Suppl.) 63 (1998) 53; PLB427 (1998) 125

- Higher Dimensional Approach
  (Domain Wall Fermion)

  H. Neuberger, NPB417 (1998) 141; PLB427 (1998) 353
  P. Hernándes, K. Jansen, M. Lüscher, NPB552 (1999) 363

Chiral Symmetry Breaking on the Lattice

- Species doubling

$$\langle \psi(k)\,\bar{\psi}(-k)\rangle = \frac{-i\gamma_\mu \frac{1}{a}\sin k_\mu a}{\sum_\nu \frac{1}{a^2}\sin^2 k_\nu a}$$

Pole appears at $k_\mu a = (0,0,0,0)$
But it also appears at $(\pi,0,0,0),\cdots,(\pi,\pi,\pi,\pi)$

- Wilson term: Lift the degeneracy due to species doublers

$$S_{\rm w} = a^4 \sum_x \bar{\psi}(x)\left\{\gamma_\mu \frac{1}{2}\left(\partial_\mu - \partial_\mu^\dagger\right) + \frac{a}{2}\left(\partial_\mu \partial_\mu^\dagger\right)\right\}\psi(x)$$

in the momentum space

$$\sum_\mu \frac{a}{2}\left(\frac{2}{a}\sin\frac{k_\mu a}{2}\right)^2 \simeq \frac{2n}{a}$$

where n is the number of π of the "zero momentum"

- Lost of manifest chiral symmetry:
$\Longrightarrow$ Inevitable due to Nielsen-Ninomiya Theorem

1. $\tilde{D}(k)$ is a periodic and analytic function of momentum k_μ (locality)
2. For small momentum $|k_\mu|a \ll \pi$, $\tilde{D}(k) \simeq i\gamma_\mu k_\mu$
3. $\tilde{D}(k)$ is invertible for all k_μ except $k_\mu = 0$ (no species doublers)
4. $\gamma_5 D + D\gamma_5 = 0$ (chiral invariance)

Renormalization Group Analysis of Chiral Symmetry Breaking

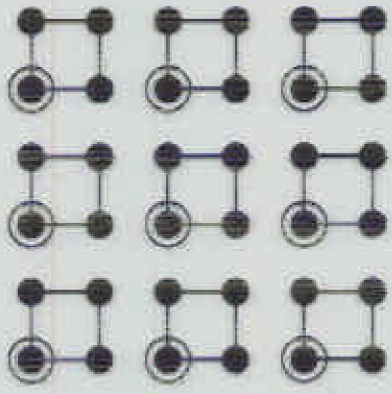
Single massless Dirac fermion for sufficiently small momentum

$$|k_\mu| \ll \pi/a$$

$\Longrightarrow$ Low energy effective action for the light mode ?!

Block Spin Transformation   Ginsparg and Wilson

$\psi'(x') = \frac{Z}{2^4} \sum_{x \in b(x')} \psi(x)$



Effective action for the blocked variables:

$$e^{-S'[\psi',\bar{\psi}']} = \int \prod_x d\psi(x) d\bar{\psi}(x)\, e^{-S_W[\psi,\bar{\psi}]} \times e^{-\alpha \sum_{x'} \left(\bar{\psi}'(x') - B(x';\bar{\psi})\right)\left(\psi'(x') - B(x';\psi)\right)}$$

Fixed point action S^*:

$$S^* = a^4 \sum_x \bar{\psi}(x) D^* \psi(x)$$

$$\gamma_5 D^* + D^* \gamma_5 = a D^* \gamma_5 R D^*, \qquad R = \frac{2}{\alpha}$$

The Ginsparg-Wilson Relation and Exact Chiral Symmetry on the Lattice

- Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

In terms of the fermion propagator $S_F = D^{-1}$:

$$\gamma_5 S_F(x,y) + S_F(x,y)\gamma_5 = \gamma_5 \delta(x,y)$$

Chiral symmetry is broken only in local contact terms !

- Such local terms would not contribute to the physical amplitudes evaluated at long-distance, $x - y \neq 0$

- Exact Chiral Symmetry

The Ginsparg-Wilson relation implies an exact symmetry of the fermion action — Lüscher

$$\gamma_5 D + D\hat{\gamma}_5 = 0$$

$$\hat{\gamma}_5 = \gamma_5 (1 - aD) \qquad (\hat{\gamma}_5)^2 = 1$$

Fermion action is invariant under the following transformation:

$$\begin{aligned} \psi(x) &\longrightarrow \psi(x) + i\alpha\, \hat{\gamma}_5 \psi(x) \\ \bar{\psi}(x) &\longrightarrow \bar{\psi}(x) + i\alpha\, \bar{\psi}(x)\gamma_5 \end{aligned}$$

Neuberger's Dirac operator

Neuberger, Neuberger-Narayanan

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right) = \frac{1}{2a}\left(1 + \gamma_5\frac{H}{\sqrt{H^2}}\right)$$

where

$$X = \left(D_{\rm w} - \frac{m_0}{a}\right), \quad H = \gamma_5 X, \quad (0 < m_0 < 2)$$

Ginsparg-Wilson relation

$$\begin{aligned} &\left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right)\gamma_5\left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right) \\ &= \gamma_5 + \gamma_5\left(\frac{H}{\sqrt{H^2}}\right)^2 + \frac{H}{\sqrt{H^2}} + \gamma_5\frac{H}{\sqrt{H^2}}\gamma_5 \\ &= \gamma_5\left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right) + \left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right)\gamma_5 \end{aligned}$$

Free Fermion

$$\tilde{D}(p) = \frac{1}{2a}\left(1 + \frac{i\gamma_\mu\bar{p}_\mu + \frac{a}{2}\hat{p}^2 - \frac{m_0}{a}}{\sqrt{\bar{p}^2 + \left(\frac{a}{2}\hat{p}^2 - \frac{m_0}{a}\right)^2}}\right)$$

$$\bar{p}_\mu = \frac{1}{a}\sin p_\mu a, \quad \hat{p}_\mu = \frac{2}{a}\sin\frac{p_\mu a}{2}$$

- $\tilde{D}(p) \simeq Z\, i\gamma_\mu p_\mu \quad (|p| << \pi\ ), \quad \tilde{D}(p) \simeq 1 \quad (|p| \simeq \pi\ )$
- Analytic periodic function in momentum p_μ for $m_0 \in (0, 2)$ (Locality)

Locality   Hernándes, Jansen and Lüscher

- Non-trivial due to inverse square root of the hermitian Wilson-Dirac operator

$$\frac{H}{\sqrt{H^2}}$$

- Eigenvalue spectrum of H^2 is closely related to the size of field strength of lattice gauge field

$$\|H^2\| \geq \left\{(1-30\epsilon)^{\frac{1}{2}} - |1-m_0|\right\}^2 > 0$$

for

$$\| 1 - U_{\mu\nu}(x) \| < \epsilon, \qquad \epsilon < \frac{1}{30}\left(1 - |1-m_0|^2\right)$$

- Exponential Bound

$$\left\| \frac{1}{\sqrt{H^2}}(x,y) \right\| < \frac{\kappa}{1-t} \exp\{-\theta|x-y|/2a\}$$

where

$$\cosh\theta = \frac{\beta-\alpha}{\beta+\alpha}, \quad t = e^{-\theta}, \quad \kappa = \sqrt{\frac{4t}{\beta-\alpha}}$$

Chiral anomaly

- Chiral anomaly due to non-trivial Jacobian:

$$\delta\left[D\psi D\bar{\psi}\right] = \left[D\psi D\bar{\psi}\right] \mathrm{Tr}\left\{\gamma_5 + \hat{\gamma}_5\right\}$$

$$\mathrm{Tr}\left\{\gamma_5 + \hat{\gamma}_5\right\} = 2\mathrm{Tr}\left\{\gamma_5\left(1 - \frac{a}{2}D\right)\right\}$$

Lüscher
Yamada-Y.K.
Suzuki,Adams,Fujikawa

Topological properties of chiral anomalies

Chirality is depending on the gauge field

$$\hat{\gamma}_5 = \hat{\gamma}_5 [\mathrm{U}_\mu], \qquad (\hat{\gamma}_5)^2 = 1$$

Under the variation of the gauge field

$$U_\mu(x) \longrightarrow U_\mu(x) + \delta U_\mu(x)$$

$$\delta\hat{\gamma}_5 \, \hat{\gamma}_5 + \hat{\gamma}_5 \, \delta\hat{\gamma}_5 = 0$$

- Chiral anomaly is topological ! $\Longrightarrow$ Index Theorem
  Hasenfratz et al.

$$\delta \, \mathrm{Tr}\hat{\gamma}_5 = 0$$

- Gauge anomaly of Weyl fermion is related to 4+2 dim. topological field! $\Longrightarrow$ Gauge anomaly cancellation
  Lüscher

$$\delta \int ds dt \mathrm{Tr} \left(\frac{1 - \hat{\gamma}_5}{2} \right) [\partial_s \hat{\gamma}_5, \partial_t \hat{\gamma}_5] = 0$$

  gauge field is assumed to be 4+2 dimensional, $U_\mu(x, s, t)$
- Crucial roles for the construction of chiral gauge theories
  Lüscher, Suzuki
  Bär-Campos, Nakayama-Y.K.

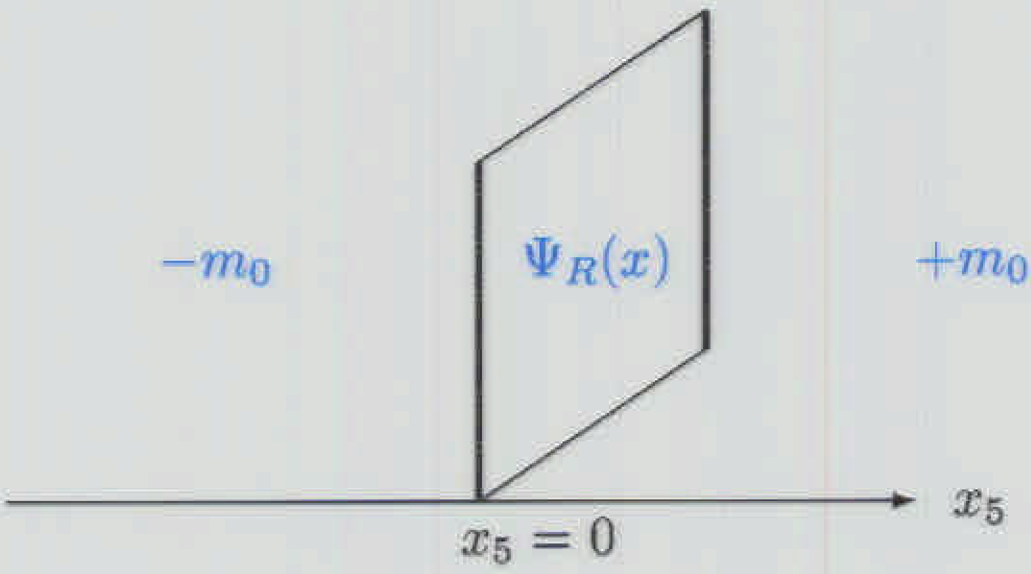
Domain wall fermion

- Five-dimensional Dirac fermion coupled to a scalar field condensate with kink-like topological defect.

$$\{\gamma_\mu D_\mu + \gamma_5 D_5 + \langle\phi(x_5)\rangle\}\,\psi(x,x_5)=0,\ \ \langle\phi(x_5)\rangle = m_0\epsilon(x_5)$$

- Chiral zero mode bounded to the domain wall at $x_5=0$

$$-\gamma_5 m_0 \delta(x_5) \in D^\dagger D \quad \Longrightarrow \quad \psi_R(x)\,e^{-m_0|x_5|}$$



- Can be implemented on the lattice with Wilson term

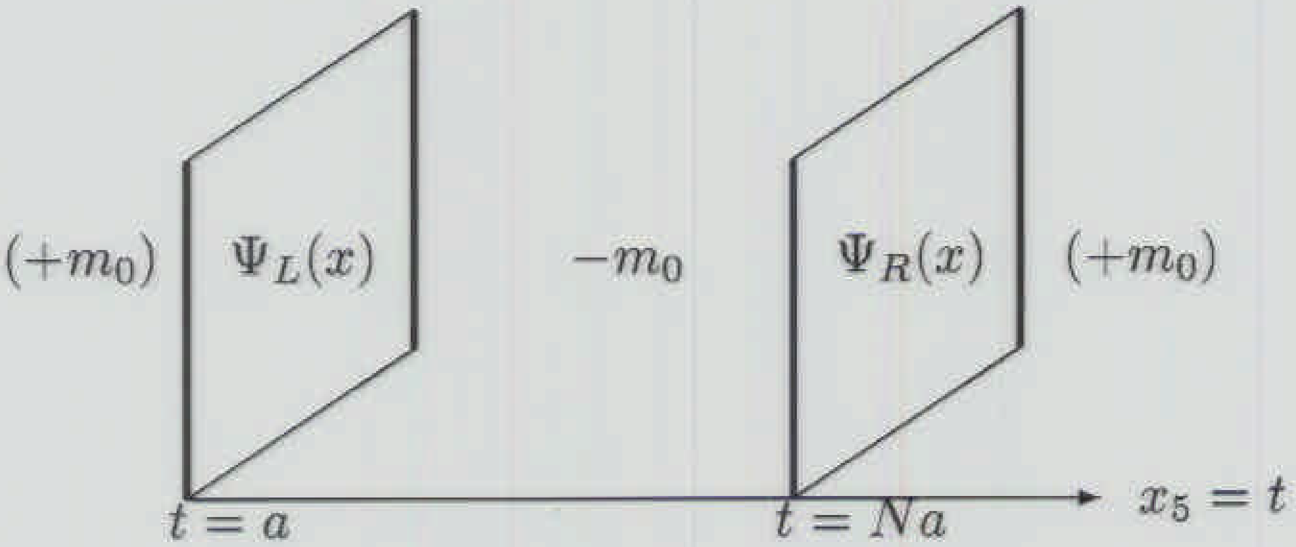
Kaplan

$$\sum_\mu \frac{2}{a}\sin^2\left(\frac{p_\mu a}{2}\right) + \frac{m_0}{a}\mathrm{sgn}(\mathrm{x}_5 - \frac{1}{2}), \quad 0 < m_0 < 2$$

Kink singularity (sign change) only for the physical mode with $|p_\mu| < \frac{\pi}{a}$

- With a finite extent of fifth-dimension, N

Shamir-Furman



- Light Dirac fermion on the walls

$$
\begin{aligned}
q(x) &= \psi_L(x,a)+\psi_R(x,Na) \\
\bar{q}(x) &= \bar{\psi}_L(x,a)+\bar{\psi}_R(x,Na)
\end{aligned}
$$

Good chiral properties of $q(x)$ and $\bar{q}(x)$ in the limit $N \to \infty$ at least for sufficiently smooth gauge fields
cf. Numerical aspects

Shamir - Furman
Aoki-Taniguchi
Neuberger-Yamada-Y.K.

$\Longleftrightarrow$ Exact chiral symmetry / Ginsparg-Wilson relation ?!

Low energy effective action for the light Dirac fermion is given by Neuberger's lattice Dirac operator !

Relation to Neuberger's lattice Dirac operator

Neuberger, Noguchi and Y.K.

- Derive low energy effective action (or propagator) of $q(x)$ and $\bar{q}(x)$ by integrating out heavy fields

$$\langle q(x)\bar{q}(y)\rangle = \frac{1}{a^5}\left({D_N}^{-1} - a\delta(x,y)\right)$$

where

$$aD_N = \frac{1}{2a}\left(1 + \gamma_5 \tanh \frac{N}{2} a_5 \widetilde{H}\right)$$

$$e^{-a_5\widetilde{H}} = T \quad \text{: transfer matrix}$$

  - The contact term subtracting the chiral symmetry breaking in the Ginsparg-Wilson relation

- Subtraction (or Decoupling) of heavy modes

$$+\frac{1}{a_5}\left(\bar{\psi}_R(x,N)\psi_L(x,1) + \bar{\psi}_L(x,1)\psi_R(x,N)\right)$$

Unit mass term for $q(x)$ and $\bar{q}(x)$ $\Rightarrow$ No light d.o.f.
$\Rightarrow$ Anti-periodic B.C.
Vranas

$$D_{5\mathrm{w}}(s,t) = \overline{D}_{5\mathrm{w}}(s,t) - P_L\frac{1}{a_5}\delta_{s,N}\delta_{t,1} - P_R\frac{1}{a_5}\delta_{s,1}\delta_{t,N}$$

$\Longrightarrow$

$$\frac{\det D_{5\mathrm{w}}}{\det \overline{D}_{5\mathrm{w}}} = \det D_N$$

- Locality

$$aD_N = 1 - \frac{1}{a_5}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}{}_{(N,N)}$$

$\Longrightarrow$ Behavior of the inverse of $\overline{D}_{5\mathrm{w}}$ $\quad (a_5 = a)$

$$a\overline{D}_{5\mathrm{w}}(x,y;p) = i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)$$

$$\left\| a^2 \overline{D}^{\dagger}_{5\mathrm{w}} \overline{D}_{5\mathrm{w}} \right\| = \left\| 4\sin^2(p/2)\left(1 - m_0 - \frac{a^2}{2}\nabla_\mu \nabla^*_\mu\right) + (aD_{\mathrm{w}} - m_0)^\dagger (aD_{\mathrm{w}} - m_0) \right\|$$

$$\left\| a^2 \overline{D}^{\dagger}_{5\mathrm{w}} \overline{D}_{5\mathrm{w}} \right\| > \left\{ (1 - 30\epsilon)^{\frac{1}{2}} - |1 - m_0| \right\}^2 > 0$$

for

$$\| 1 - U_{\mu\nu}(x) \| < \epsilon, \qquad 0 < \epsilon < \frac{1 - |1 - m_0|^2}{30}$$

cf. Hernándes, Jansen and Lüscher

5 dim. Bounds for Locality and Chirality

$$\left\| \left\{ a^2 \overline{D}^{\dagger}_{5\mathrm{w}} \overline{D}_{5\mathrm{w}} \right\}^{-1} (x,s;y,t) \right\| \le C \exp\left\{ -\frac{\tilde{\theta}}{2a} d_5(x,s;y,t) \right\}$$

where

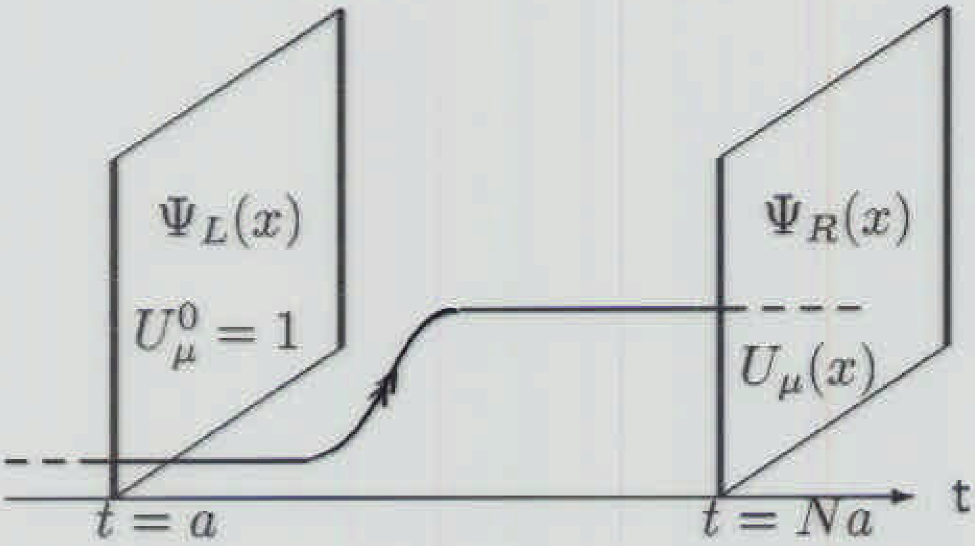
$$C = \frac{4t}{\tilde{\beta} - \tilde{\alpha}} \left(\frac{1}{1-t} \frac{d_5(x,s;y,t)}{2a} + \frac{t}{(1-t)^2} \right)$$

$$t = e^{-\tilde{\theta}}, \qquad \cosh \tilde{\theta} = \frac{\tilde{\beta} + \tilde{\alpha}}{\tilde{\beta} - \tilde{\alpha}}$$

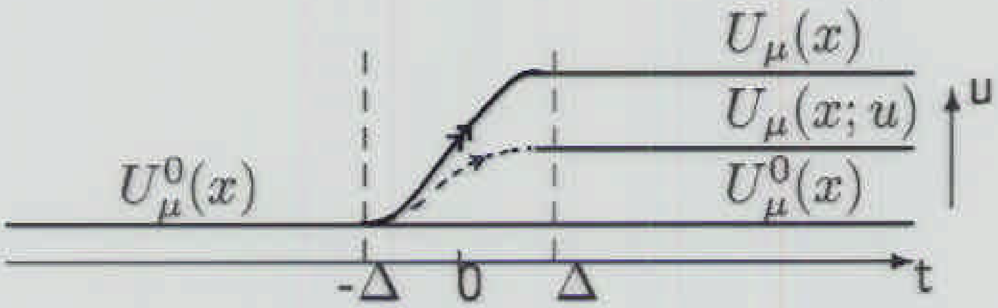
$$d_5(x,s;y,t) = |x - y| + \min(|s - t|, N - |s - t|)$$

Domain wall fermion and Chiral gauge theory

- Chiral coupling of gauge field to Ψ_R
  $\Longrightarrow$ Interpolating five-dimensional gauge field !



- Dependence on the interpolation path, Gauge anomaly
  $\Longrightarrow$Gauge current flow
  Variation of determinant of domain wall fermion with respect to gauge field (the parameter u)



$$\frac{d}{du}\lim_{T\to\infty}\ln\det\left(D_{5\mathrm{w(T)}} - \frac{m_0}{a}\right)$$
$$= \lim_{T\to\infty}\mathrm{Tr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w}(\infty)}\frac{1}{D_{5\mathrm{w}(\infty)} - \frac{m_0}{a}} + \mathrm{Tr}_x\,P_L\frac{d}{du}D\,\frac{1}{D}$$

Aoyama-Y.K.

- Current on 4 dim. wall
  $\Longleftrightarrow$ Effective action for Weyl fermion / Ginsparg-Wilson rel.
  - Weyl fermion

$$\hat{\gamma}_5 \psi_R(x) = +\psi_R(x), \quad \bar{\psi}_R(x)\gamma_5 = -\bar{\psi}_R(x)$$

  - Effective action

$$\frac{d}{du}\Gamma_{\text{eff}}[U_\mu] = \mathrm{Tr}_x\, P_L \frac{d}{du} D \frac{1}{D} + \cdots$$

cf. Lüscher, Suzuki

- Current in 5 dim. bulk = Chern-Simons current

$$\mathrm{Tr}\, \frac{d}{du} D_{5\mathrm{w}} \frac{1}{D_{5\mathrm{w}} - \frac{m_0}{a}} = \sum_{z=x,t} \frac{d}{du} U_\mu \cdot U_\mu^{-1} J_\mu(z)$$

$$\int_0^1 du \sum_z \frac{d}{du} U_\mu \cdot U_\mu^{-1} J_\mu(z)$$

$\Longrightarrow$ Chern-Simons term plus local counter term
- integrability
  Independence on the five-dimensional interpolation path
- gauge anomaly
  exact gauge invariance

Conclusion

- Domain wall fermion can produce chiral fermions bounded to the 4D walls, under certain conditions for lattice gauge fields
- Low energy effective theories of these chiral fermions can be constructed so that both locality and chiral symmetry is maintained by virtue of the Ginsparg-Wilson relation
- Chiral gauge-coupling can be introduced for these chiral fermions, through the interpolating five-dimensional gauge fields
- A five-dimensional local counter term should take account of the integrability and the gauge invariance