Structure Functions on the Lattice

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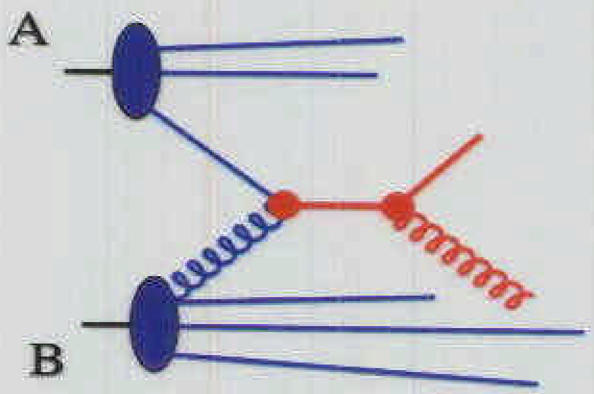
in place of Marco Guagnelli

- **Introduction**
- **the stony way**
- **the answers: QCD versus experiment**
  at least in valence quark approximation
- **Conclusion**

Parton distribution functions

the probability to find a parton carrying a fraction x_A of a hadron's momentum in the process



$$\text{hadron } A + \text{hadron } B \rightarrow \text{jet} + X$$

is

$$f_a(x_A)dx_A$$

where

$$f_a(x_A)$$

is the **"parton distribution function"**

(correspondingly a "gluon distribution function" may be defined)

Moments ↔ local operators

the lattice can contribute because instead of the parton distribution functions themselves, moments of them can be considered

$$M_a^{(n)}(\mu) = \int_0^1 dx x^{n-1} f_a(x,\mu)$$

$$n = 1, 2, \cdots$$

and the moments can be related to expectation values of *local operators* $\mathcal{O}_a^n$

$$M_a^{(n)}(\mu) = \langle \mathcal{O}_a^n \rangle$$

where

$$\{p^{\mu_1} \cdots p^{\mu_n}\}_{\mathrm{TS}} \langle \mathcal{O}_a^n \rangle =$$

$$\langle p | \bar{\psi}_a(x) \{\gamma^{\mu_1} i D^{\mu_2} \cdots i D^{\mu_n}\}_{\mathrm{TS}} \psi_a(x) | p \rangle$$

- $\langle p|$ state vector of a hadron with momentum p
- D^μ covariant derivative (gauge invariance)
- TS means to take traceless symmetric part

The goal

Experiment gives for the average momentum of a parton (≡ the second moment)ina pion and for a non-singlet (NS) operator

$$\langle x\rangle(\mu=2.4\mathrm{GeV}) \;=\; 0.23\pm 0.02 \quad \overset{?}{\Leftarrow} \; \mathrm{QCD}$$

$$\langle x\rangle_{\overline{\mathrm{MS}}}(\mu=2.4\mathrm{GeV}) \;=\; \cdots \leftarrow \mathrm{non-perturbatively}$$

non-peturbative ⇒ lattice

pioneering work

QCDSF collaboration

Best, Göckeler, Horsley, Ilgenfritz, Perlt, Rakow, Schäfer, Schierholz, Schiller, Schramm

conventional approach: find

$$\langle x\rangle_{\overline{\mathrm{MS}}}(\mu=2.4\mathrm{GeV}) = 0.27(1)$$

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? contradiction with experiment ?

drawbacks of this calculation

a) no continuum extrapolation

b) perturbative renormalization factor

to clarify the importance of a) and b)

⇒ different approach:
Bucarelli, Guagnelli, Jansen, Palombi, Petronzio, Shindler
hep-lat/9808005, 9809009, 9901016, 9903012, 00?????

assume a fictitious scheme (the SF-scheme)

modest but ambitious aim: compute

$$\langle x\rangle_{\mathrm{SF}}(\mu_0) = \lim_{a\to 0} \lim_{\mathrm{chiral}} \frac{\langle\pi|O_{\mathrm{NS}}^{n=2}|\pi\rangle}{Z^{\mathrm{SF}}(1/\mu_0)}$$

μ_0 chosen such that the calculation is *"easy"*

⇒ μ_0 low scale

assume now that we can compute *non-perturbatively*

the running matrix element $\langle x \rangle_{\mathrm{SF}}(\mu)$

$\Rightarrow$ for small enough coupling $\bar{g}$, define the *renormalization group invariant matrix element*

$$O_{\mathrm{INV}}^{\mathrm{ren}} = O_{\mathrm{SF}}^{\mathrm{ren}}(\mu) \cdot f^{\mathrm{SF}}(\bar{g}^2(\mu))$$

$$f^{\mathrm{SF}}(\bar{g}^2(\mu)) = (\bar{g}^2(\mu))^{-\gamma_0/2b_0} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

$\gamma(g)$: anomalous dimension function

$\beta(g)$: β-function
(in SF scheme)

knowing O^{ren} we can get "easily" the matrixelement in a desired scheme

$$\langle x \rangle^{\mathrm{SF}} = O_{\mathrm{INV}}^{\mathrm{ren}} f^{\mathrm{SF}}(\bar{g}^2(\mu))$$

$$\langle x \rangle^{\overline{\mathrm{MS}}} = O_{\mathrm{INV}}^{\mathrm{ren}} f^{\overline{\mathrm{MS}}}(\bar{g}^2(\mu))$$

$$\langle x \rangle^{\mathrm{myprefered}} = O_{\mathrm{INV}}^{\mathrm{ren}} f^{\mathrm{myprefered}}(\bar{g}^2(\mu))$$

How to get $O^{\rm ren}_{\rm INV}$?

$$
\begin{aligned}
O^{\rm ren}_{\rm INV} &= O^{\rm ren}_{\rm SF}(\mu)\cdot f^{\rm SF}(\bar{g}^2(\mu)) \\
&= \frac{\langle\pi|O_{\rm NS}|\pi\rangle}{Z^{\rm SF}(1/\mu)}\cdot f^{\rm SF}(\bar{g}^2(\mu)) \\
&= \frac{\langle\pi|O_{\rm NS}|\pi\rangle}{Z^{\rm SF}(1/\mu_0)}\cdot \underbrace{\frac{Z^{\rm SF}(1/\mu_0)}{Z^{\rm SF}(1/\mu)}}_{\equiv\sigma(\mu/\mu_0,\bar{g}(\mu))} \cdot f^{\rm SF}(\bar{g}^2(\mu)) \\
&\equiv O^{\rm ren}_{\rm SF}(\mu_0)\underbrace{\sigma(\mu/\mu_0,\bar{g}(\mu))\cdot f^{\rm SF}(\bar{g}^2(\mu))}_{\equiv\mathfrak{S}^{\rm UV}_{\rm INV}(\mu_0)}
\end{aligned}
$$

- $O^{\rm ren}_{\rm SF}(\mu_0)$ renormalized matrix element $\rightarrow$ only to be computed once
- $\sigma(\mu/\mu_0,\bar{g}(\mu))$ *step scaling function*
- $\mathfrak{S}^{\rm UV}_{\rm INV}(\mu_0)$ (ultraviolett)
  renormalization group invariant step scaling function

$$
\boxed{O^{\rm ren}_{\rm INV} = O^{\rm ren}_{\rm SF}(\mu_0)\,\mathfrak{S}^{\rm UV}_{\rm INV}(\mu_0)}
$$

SF≡Schödinger functional renormalization scheme

→ finite volume scheme, identifying the scale with the box length

$$\mu = L^{-1}$$

renormalization condition:

$$\langle SF|O^R(\mu = 1/L)|SF\rangle = \langle SF|O^{\rm tree}|SF\rangle$$

with $|SF\rangle$ a Schödinger functional state, i.e. a classical quark at the (time) boundary with an external momentum $\mathbf{p}$

relation between bare operator and renormalized operator through the normalization constant

$$O^R(\mu) = Z(1/\mu)^{-1} O^{\rm bare}(1/L)$$

and hence

$$O^R(\mu) = Z(1/\mu)^{-1} Z(L) O^{\rm tree}$$

in perturbation theory

$$Z(1/\mu) = 1 - \bar{g}^2(\mu)\left[\gamma^{(0)} \ln(\mu) + B_0\right]$$

with $\gamma^{(0)}$ the anomalous dimension

and B_0 the constant part

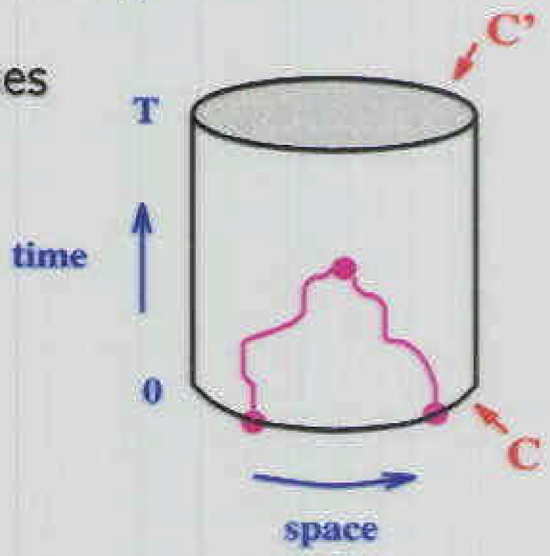
to extract matrixelements O we have to compute fermion correlation functions in the Schrödinger functional:

A correlation function $f_{\mathcal{O}}$ of some operator $\mathcal{O}$ at a distance x_0/a

$$f_{\mathcal{O}}(x_0/a) = \sum_{y,z} < \mathcal{O}(x)\bar{\zeta}(y)\Gamma\tau^3\zeta(z) >$$

ζ, $\bar{\zeta}$ are boundary quark fields at $x_0 = 0$

Γ is a suitable set of γ matrices



$\rightarrow$ need normalization of boundary fields

$$f_1(x_0) = -\frac{1}{L^6}\sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z}} \frac{1}{3}\langle\bar{\zeta}'(\mathbf{u})\Gamma\frac{1}{4}\tau^3\zeta'(\mathbf{v}\bar{\zeta}(\mathbf{y})\Gamma\frac{1}{4}\tau^3\zeta(\mathbf{z})\rangle$$

definition of the normalization constant

$$Z(L) = \frac{\bar{Z}(L)}{\sqrt{f_1(L)}}$$

and we have the step scaling functions

$$\sigma_{\bar{Z}} = \frac{\bar{Z}(2L)}{\bar{Z}(L)},\ \sigma_{f_1} = \frac{\sqrt{f_1}(2L)}{\sqrt{f_1}(L)},\ \sigma_Z = \frac{Z(2L)}{Z(L)}$$

the normalization constant is a function of *dimensionless quantities*

$$Z = Z(pL, x_0/L, a/L)$$

if we only want to change the scale, identified as $\mu^{-1} = L$, we have to keep

$pL,\ x_0/L$ fixed

while tuning $a/L \to 0$ to reach the continuum limit:

- $pL = 2\pi$ (lowest momentum)
- $x_0/L = 1/4$

the numerical simulations are performed using

- Wilson fermions
- full $O(a)$ non-perturbative improved Wilson fermions

for complete $O(a)$ improvement we would need also the operator to be improved, giving the form

$$\begin{aligned} \mathcal{O}_{\mathrm{impr}} &= c_1 \bar{\psi}\gamma_\mu D_\nu \psi + c_2 \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu} \\ &+ c_3 \bar{\psi}\{D_\mu, D_\nu\} + c_4 \partial_\lambda\left(\bar{\psi}\sigma_{\mu\lambda}D_\nu\right) \end{aligned}$$

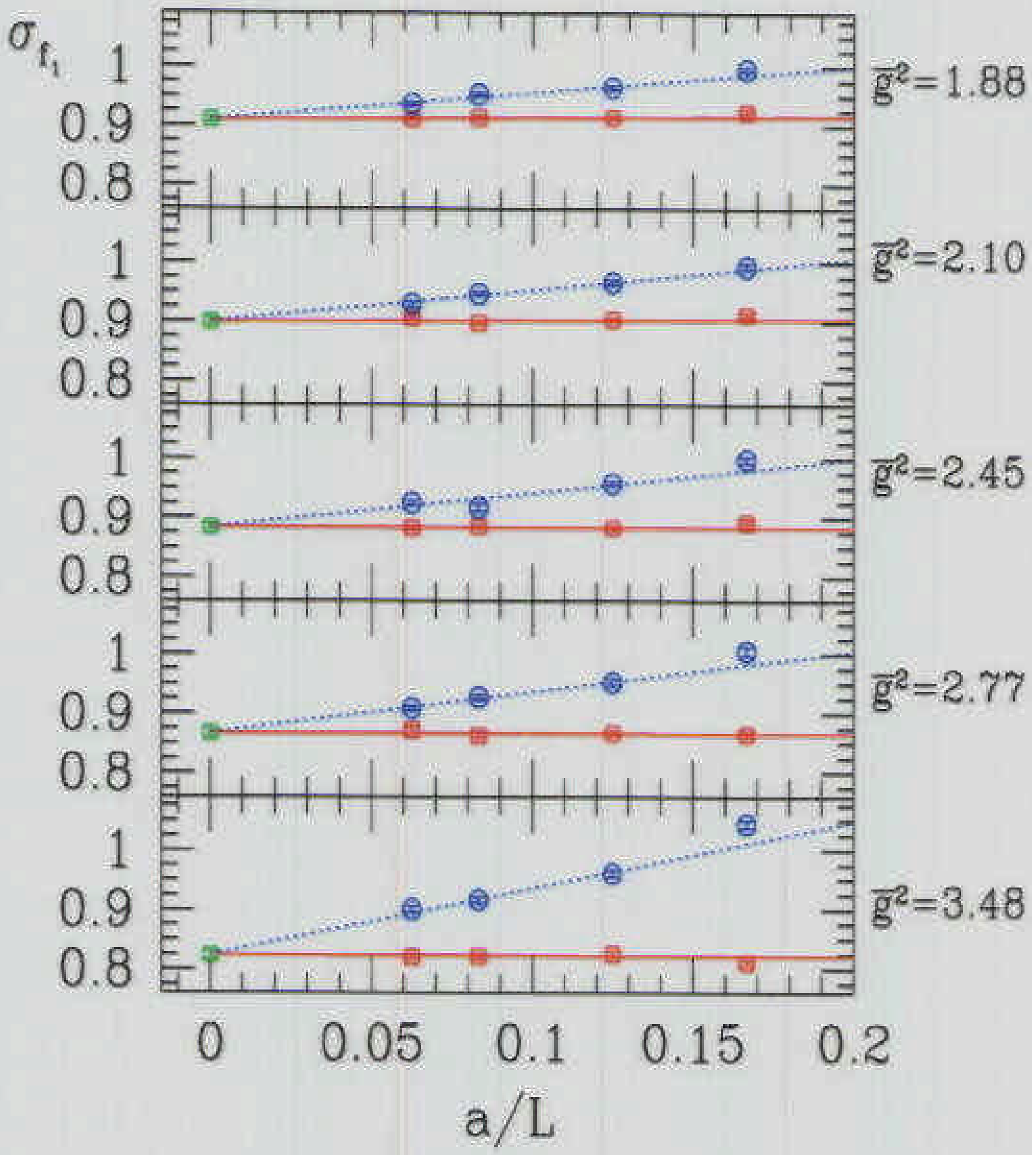
QCDSF heplat/9711007

$\Rightarrow$ complicated operator and difficult to determine the coefficients $c_1, \cdots, c_4$ non-perturbatively

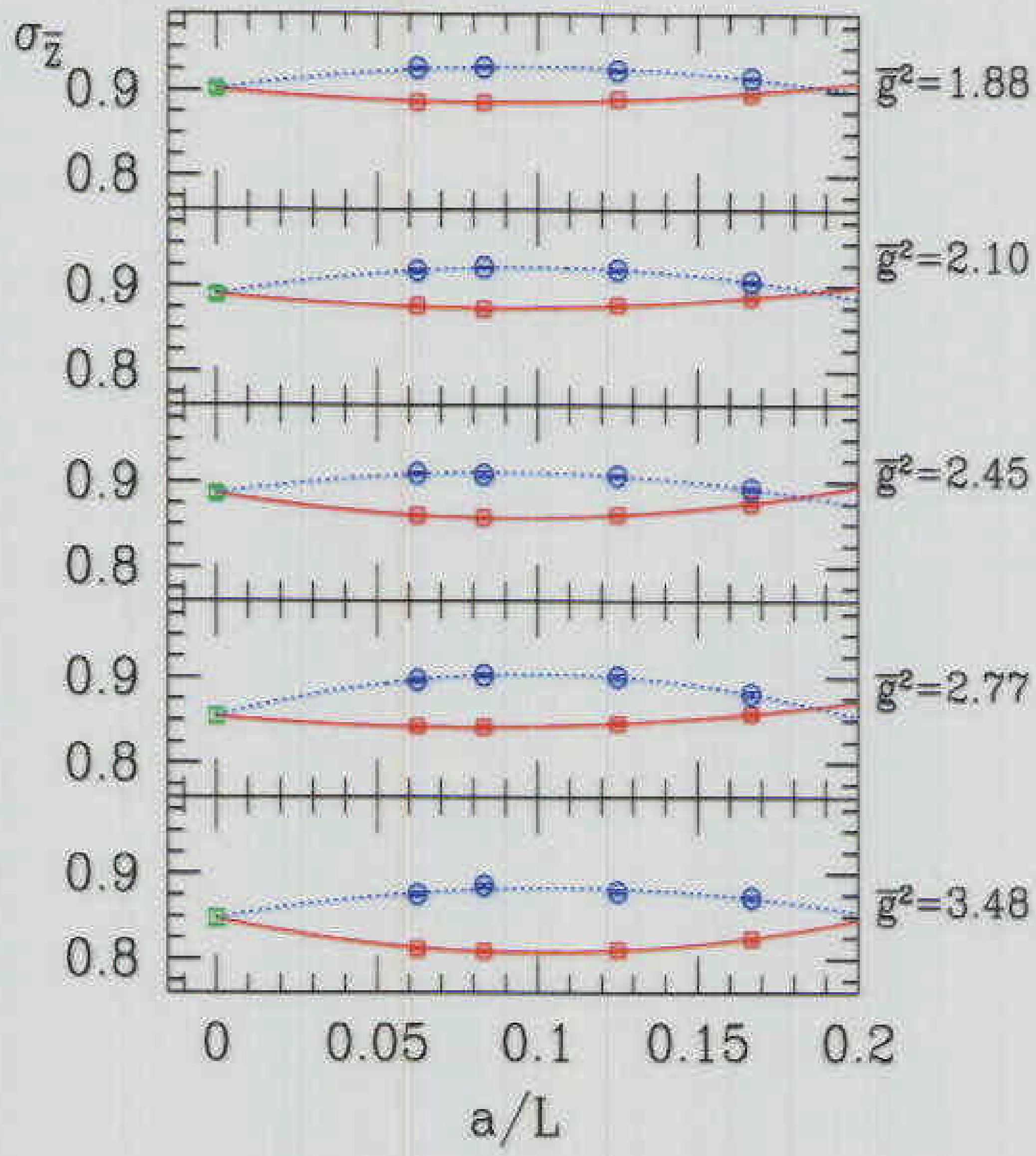
$\rightarrow$ attempt to first only improve the action

$\rightarrow$ provides a check on the continuum extrapolation of σ

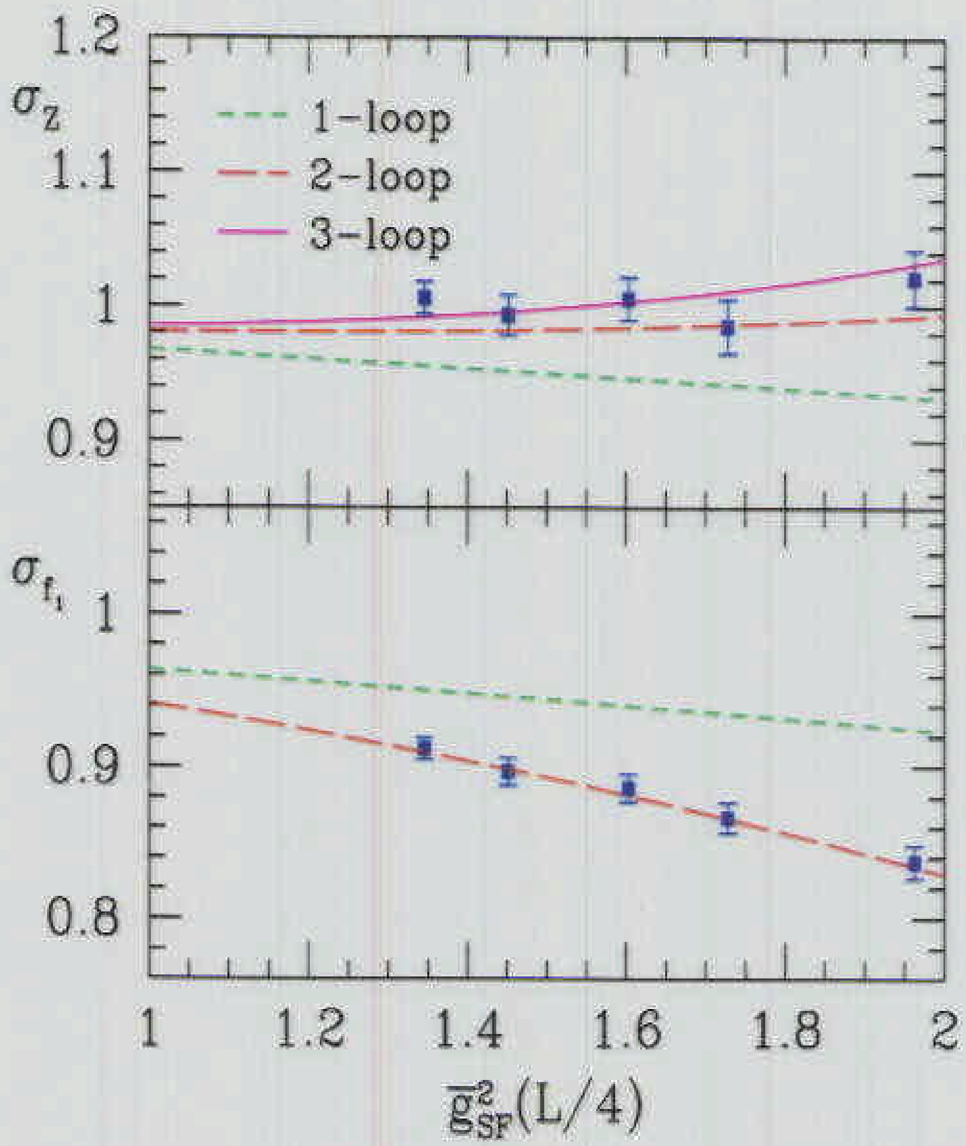
Combined fit σ_{f_1}



Combined fit $\sigma_{\bar{Z}}$



Continuum step scaling functions



choice of $\bar{g}^2_{\mathrm{SF}}(\mu^{-1} = L/4)$ from the definition of the operator taken at $x_0 = L/4$

the invariant step scaling function

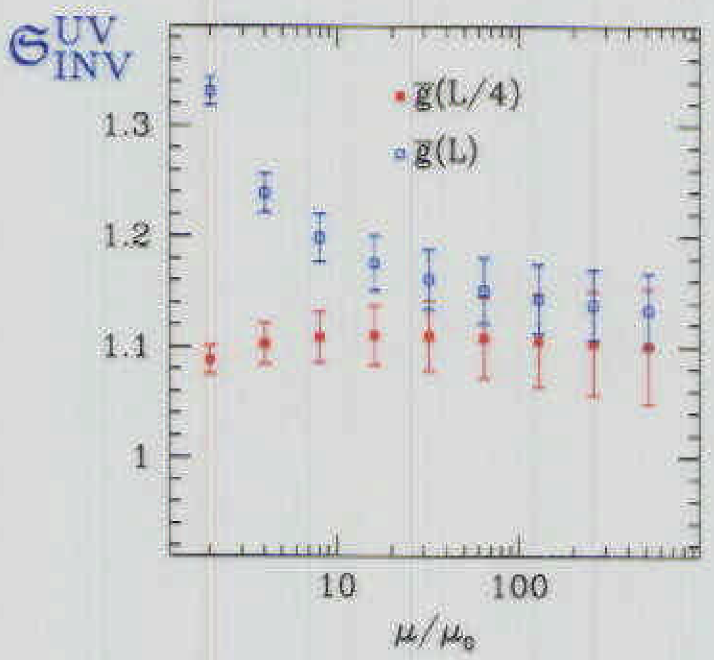
In the perturbative region we can define the renormalization group invariant step scaling function $\mathfrak{S}^{\mathrm{UV}}_{\mathrm{INV}}(\mu_0)$

$$\mathfrak{S}^{\mathrm{UV}}_{\mathrm{INV}}(\mu_0) = \sigma(\mu/\mu_0, \bar{g}^2(\mu_0)) \cdot f(\bar{g}^2(\mu))$$

with

$$f(\bar{g}^2(\mu)) = (\bar{g}^2(\mu))^{-\gamma_0/2b_0} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

with the β and γ functions taken up to 3-loops

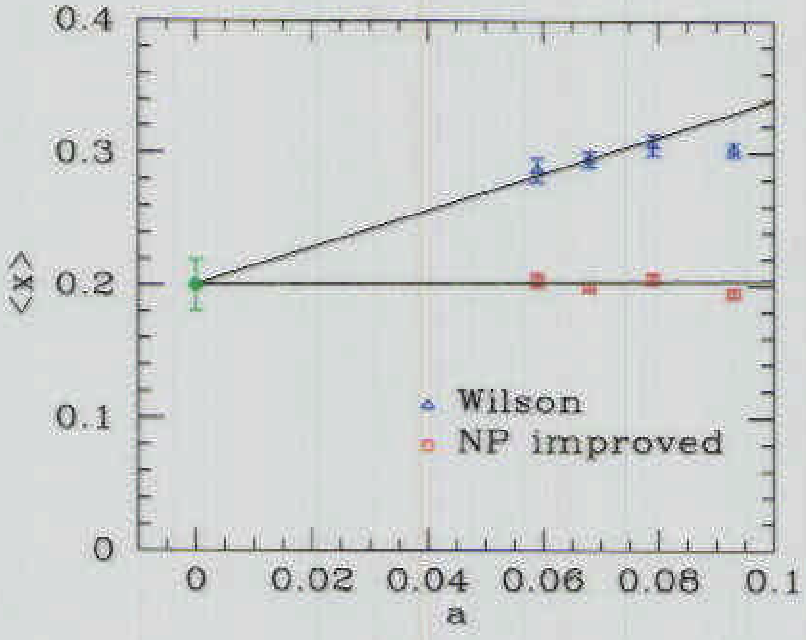


$$\mathfrak{S}^{\mathrm{UV}}_{\mathrm{INV}} = 1.11(4)$$

continuum limit of renormalized matrix element

$$\lim_{a\to 0} \left.\frac{<\pi|O_2|\pi>}{Z_{O_2}(L_0)}\right|_{m_q=0}$$

- continuum limit while kepping $\mu_0^{-1} = L_0 = 0.7 \cdot r_0$ fixed $(r_0 = 0.5fm)$
- use two action method to check for universal continuum limit



$$\langle x\rangle_{\rm SF}(\mu_0) = \lim_{a\to 0}\lim_{\rm chiral}\frac{\langle\pi|\mathcal{O}_{\rm NS}^{n=2}|\pi\rangle}{Z^{\rm SF}(1/\mu_0)} = 0.20(2)$$

the result

- main result:
  renormalization group invariant matrix element

$$O_{\mathrm{INV}}^{\mathrm{ren}} = O_{\mathrm{SF}}^{\mathrm{ren}}(\mu_0)\mathfrak{S}_{\mathrm{INV}}^{\mathrm{UV}}(\mu_0) = 0.20 \cdot 1.11$$

↑

μ_0 dependence cancelled

- find sizable lattice artefacts:
  $\langle x \rangle (a = 0.093) = 0.30 \rightarrow \langle x \rangle (a = 0) = 0.20$
- non-perturbative renormalization → 10-15% effect
- summing it all up, however,

$$\langle x \rangle^{\mathrm{experiment}}(\mu = 2.4\mathrm{GeV}) = 0.23 \pm 0.02$$

$$\langle x \rangle_{\overline{\mathrm{MS}}}^{\mathrm{quenched}}(\mu = 2.4\mathrm{GeV}) = 0.30(3)$$

⇒ discrepancy remains!
(through conspiration of effects, however, ...)

Conclusion

computation of average momentum
≡ 2nd momentum of parton distribution function

- non-perturbative renormalization
- continuum extrapolation

⇒ no *systematic* errors left

♦ catch: quenched approximation

find discrepancy with experimental number

⇒ further work:

→ unquenching

→ Roman window technique (still in its infancy)

→ further moments (as computed by QCDSF)