Macroconstraints from Microsymmetries

based on

Dynamics of Multiparticle Systems with non − Abelian Symmetry *

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Ingredients:

0. An internal symmetry group G.

I. A multiparticle system which transforms under some representation(s) of the group G − *macrosymmetry*.

II. G-invariant local interaction − *microsymmetry*.

III. A set of distribution functions $f^{(\cdot)}(\Gamma, \vec{r}, t)$ satisfying *evolution equations*.

Problem:

- Is macrosymmetry preserved during a time evolution of the system?

Answer:

A. For abelian symmetries:

microsymmetry = macrosymmetry

B. For nonabelian symmetries:

macrosymmetry$\Longrightarrow$subsidiary conditions on evolution equations

Representation are denoted as α_i with corresponding dimensions $d(\alpha_i)$.

$f_{(\zeta)}^{(\alpha_i,\nu_i)}(\Gamma, \vec{r}, t)$ – a distribution function of the particle which belongs to the multiplet α_i

Members of this multiplet are numbered by indexes ν_i ($\nu_i = 1, \ldots, d(\alpha_i)$) which correspond to given values of charges related to the symmetry group.

A number of particles of the specie $\{\alpha, \nu_\alpha, \zeta\}$

$$N_{\nu_\alpha;(\zeta)}^{(\alpha)}(t) = \int dV d\Gamma f_{(\zeta)}^{(\alpha,\nu_\alpha)}(\Gamma, \vec{r}, t)\,;$$

State vectors in particle number representation:

$$\left| N_{\nu_{\alpha_1}}^{(\alpha_1)}, \ldots, N_{\nu_{\alpha_n}}^{(\alpha_n)} \right\rangle .$$

Can be decomposed into direct sum of irreducible representations Λ_k with states

$$\left| \Lambda_k, \lambda_{\Lambda_k}; \mathcal{N} \right\rangle$$

$$\mathcal{N} = \sum_k N_{\nu_{\alpha_k}}^{(\alpha_k)} \ ;$$

$$\left| N_{\nu_{\alpha_1}}^{(\alpha_1)}, \ldots, N_{\nu_{\alpha_n}}^{(\alpha_n)} \right\rangle =$$

$$\sum_k^{\oplus} \sum_{\xi_{\Lambda_k}}^{\oplus} |\Lambda_k, \lambda_{\Lambda_k}; \mathcal{N}; \xi_{\Lambda_k}\rangle \, a^{\Lambda,\lambda_\Lambda}_{\{N_{\nu_{\alpha_1}}^{(\alpha_1)}, \ldots, N_{\nu_{\alpha_n}}^{(\alpha_n)}\}}(\xi_{\Lambda_k}; \Gamma)$$

ξ_Λ — degeneracy parameters.

An average weight

$$\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} =$$

$$\frac{\sum\limits_{\xi_\Lambda} |a^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda;\Gamma)|^2}{\sum\limits_{N^{(\alpha_1)}_{\nu_{\alpha_1}}+\cdots+N^{(\alpha_n)}_{\nu_{\alpha_n}}=\mathcal{N}} \sum\limits_{\xi_\Lambda} |a^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda;\Gamma)|^2};$$

Statistical hypothesis: All weights

$$\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}}$$

can be calculated only from the symmetry properties.

A projection operator on the subspace spanned by all states transforming under representation Λ:

$$\mathcal{P}^{\Lambda}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle \;=$$

$$\sum_{\xi_\Lambda}^{\oplus}|\Lambda,\lambda_\Lambda;\xi_\Lambda\rangle\,\mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda)\,;$$

$$\mathcal{P}^{\Lambda}=d(\Lambda)\int_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)U(g)$$

$$U(g)\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle \;=$$

$$\sum_{\nu_1^{(1)},\ldots,\nu_n^{(N\nu_n)}} D^{(\alpha_1)}_{\nu_1^{(1)}\nu_1}\cdots D^{(\alpha_1)}_{\nu_1^{(N\nu_1)}\nu_1}\cdots D^{(\alpha_n)}_{\nu_n^{(1)}\nu_n}\cdots D^{(\alpha_n)}_{\nu_n^{(N\nu_n)}\nu_n}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle\;\;;$$

The statistical hypothesis:

$$\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} = \left\| \mathcal{P}^\Lambda \left| N^{(\alpha_1)}_{\nu_{\alpha_1}}, \ldots, N^{(\alpha_n)}_{\nu_{\alpha_n}} \right\rangle \right\|^2$$

$$= \left\langle N^{(\alpha_1)}_{\nu_{\alpha_1}}, \cdots, N^{(\alpha_n)}_{\nu_{\alpha_n}} \right| \mathcal{P}^\Lambda \left| N^{(\alpha_1)}_{\nu_{\alpha_1}}, \ldots, N^{(\alpha_n)}_{\nu_{\alpha_n}} \right\rangle$$

$$= \sum_{\xi_\Lambda} |\mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda)|^2$$

$$= \mathcal{A}^{\{\mathcal{N}\}} d(\Lambda) \int\limits_G d\mu(g) \bar{\chi}^{(\Lambda)}(g) [D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}} \cdots [D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}$$

$\mathcal{A}^{\{\mathcal{N}\}}$ — permutation factor.

For particles of the kind $\{\alpha, \zeta\}$ there is the permutation factor

$$\mathcal{A}^{\alpha}_{(\zeta)} = \frac{\mathcal{N}^{(\alpha)}_{(\zeta)}!}{\prod\limits_{\nu_\alpha} \mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}!} \; ;$$

The permutation factor $\mathcal{A}^{\{\mathcal{N}\}}$ is a product of all "partial" factors

$$\mathcal{A}^{\{\mathcal{N}\}} = \prod_j \prod_{\zeta_j} \mathcal{A}^{\alpha_j}_{(\zeta_j)}$$

When macrosymmetry is preserved then all weights should be constant.

$$\frac{d}{dt}\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\dots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} = 0\,;$$

The time derivative of the normalization factor $\mathcal{A}^{\{\mathcal{N}\}}$?

Analytic continuation from integer to continuous values of variables $N_{\nu_{\alpha_n}}^{(\alpha_n)}$.

$$\psi(x) = \frac{d\,\log\Gamma(x)}{d\,x}\,;$$

$$\frac{d\,\mathcal{A}^{\{\mathcal{N}\}}}{dt} = \mathcal{A}^{\{\mathcal{N}\}} \sum_{j} \sum_{\zeta_j}$$

$$\left[\frac{d\mathcal{N}_{(\zeta_j)}^{(\alpha_j)}}{dt} \psi(\mathcal{N}_{(\zeta)}^{(\alpha)} + 1) - \sum_{\nu_{\alpha_j}} \frac{d\mathcal{N}_{\nu_{\alpha_j};(\zeta_j)}^{(\alpha_j)}}{dt} \psi(\mathcal{N}_{\nu_\alpha;(\zeta)}^{(\alpha)} + 1) \right]$$

Generalized Vlasov - Boltzman kinetic equations:

$$\frac{\partial f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)}{\partial t}+\vec{v}\cdot\nabla f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)=$$

$$\sum_{\alpha_j,\alpha_k,\alpha_l}\ \sum_{\nu_j,\nu_k,\nu_l}\ \sum_{\zeta_j,\zeta_k,\zeta_l}\int d\Gamma_j d\Gamma_k d\Gamma_l\,\mathcal{W}^{(\zeta_i,\zeta_j;\zeta_k,\zeta_l)}_{\nu_i\nu_j;\nu_k\nu_l}(\Gamma_k,\Gamma_l;\Gamma_j,\Gamma_i)$$

$$\Big[\mathcal{F}^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)\mathcal{F}^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)f^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)f^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)$$
$$-\mathcal{F}^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)\mathcal{F}^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)f^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)\Big]\ ;$$

The rates of particle number change from the integrated Boltzmann kinetic equation:

$$\frac{d\,N_{\nu_{\alpha_i}}^{(\alpha_i)}}{dt} = \sum_{\alpha_j,\alpha_k,\alpha_l}\sum_{\nu_j,\nu_k,\nu_l}\sum_{\zeta_j,\zeta_k,\zeta_l}\times$$

$$\int dV d\Gamma_j d\Gamma_k d\Gamma_l d\Gamma_i \,\mathcal{W}_{\nu_i\nu_j;\nu_k\nu_l}^{(\zeta_i,\zeta_j;\zeta_k,\zeta_l)}(\Gamma_k,\Gamma_l;\Gamma_j,\Gamma_i)\times$$

$$\Big[\mathcal{F}_{(\zeta_i)}^{(\alpha_i,\nu_i)}(\Gamma_i,\vec{r},t)\mathcal{F}_{(\zeta_j)}^{(\alpha_j,\nu_j)}(\Gamma_j,\vec{r},t)f_{(\zeta_k)}^{(\alpha_k,\nu_k)}(\Gamma_k,\vec{r},t)f_{(\zeta_l)}^{(\alpha_l,\nu_l)}(\Gamma_l,\vec{r},t)$$

$$-\mathcal{F}_{(\zeta_k)}^{(\alpha_k,\nu_k)}(\Gamma_k,\vec{r},t)\mathcal{F}_{(\zeta_l)}^{(\alpha_l,\nu_l)}(\Gamma_l,\vec{r},t)f_{(\zeta_i)}^{(\alpha_i,\nu_i)}(\Gamma_i,\vec{r},t)f_{(\zeta_j)}^{(\alpha_j,\nu_j)}(\Gamma_j,\vec{r},t)\Big]\ ;$$

These are necessary conditions for the preserving of an internal symmetry on the macroscopic level.

The case of abelian symmetry does not lead to new results. New results appear only for nonabelian symmetries.

Example

$SU(2)$ symmetry. Particles transform under spinor ($\mathbf{\frac{1}{2}}$) and vector ($\mathbf{1}$) representations.

Macrosystem is a singlet state.

A "chemical composition" of the state is:

$$n_n,\ n_p,\ n_-,\ n_0,\ n_+$$

$$D^{(1/2)}_{mm}(\alpha,\beta,\gamma) = e^{im(\alpha+\gamma)}\cos\frac{\beta}{2}\ ;$$

where $m = \pm\frac{1}{2}$

$$\begin{aligned} D^{(1)}_{\pm 1,\pm 1}(\alpha,\beta,\gamma) &= \frac{1}{2}e^{\pm im(\alpha+\gamma)}(1+\cos\beta) \\ D^{(1)}_{0,0}(\alpha,\beta,\gamma) &= \cos\beta \end{aligned}$$

The Haar measure:

$$\int d\mu(g) = \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta\ ;$$

The weight of the singlet state:

$$\frac{1}{\mathcal{A}^{\{\mathcal{N}\}}}\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} =$$

$$(-1)^{n_0}\frac{1}{\mathcal{R}+1}\,{}_2F_1(-n_0,\mathcal{R}+1,\mathcal{R}+2;2)\ ;$$

$$\mathcal{R} = n_n + n_p + 2n_- + 2n_+\ .$$

A permutation factor:

$$\mathcal{A}^{\{\mathcal{N}\}} = \frac{(n_- + n_0 + n_+)!(n_n + n_p)!}{n_-!n_0!n_+!n_n!n_p!}\ ;$$

The nonzero values of the weight only for

$$n_n - n_p + 2n_- - 2n_+ = 0\ ;$$

$$0 = \frac{d}{dt}\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}$$

$$= 2\frac{\partial \tilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial \mathcal{R}}\left(\frac{dn_n}{dt} + 2\frac{dn_-}{dt}\right) +$$

$$\frac{\partial \tilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial n_0}\frac{dn_0}{dt} + \frac{d \log \mathcal{A}^{(\mathcal{N})}}{dt}\tilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}} \ ;$$

Corresponding coefficients are

$$\frac{\partial \tilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial n_0} = -\pi\frac{\sin n_0\pi}{\Gamma(-n_0)}\sum_{i=0}^{\infty}\frac{\Gamma(-n_0+i)}{\mathcal{R}+1+i}\frac{2^i}{i!}+$$

$$\frac{\cos n_0\pi}{\Gamma(-n_0)}\sum_{i=0}^{\infty}\frac{\Gamma(-n_0+i)[\psi(-n_0)-\psi(-n_0+i)]}{\mathcal{R}+1+i}\frac{2^i}{i!} =$$

$$(-1)^{n_0}\sum_{i=1}^{n_0}\binom{n_0}{i}\frac{[\psi(1+n_0)-\psi(1+n_0-i)]}{\mathcal{R}+1+i}(-2)^i \ ;$$

$$\frac{\partial \tilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial \mathcal{R}} = (-1)^{n_0+1}\sum_{i=0}^{n_0}(-2)^i\binom{n_0}{i}\frac{1}{(\mathcal{R}+1+i)^2}$$

and

$$\frac{d\ \log \mathcal{A}^{(\mathcal{N})}}{dt} = \frac{dn_n}{dt}\left[\psi(n_N+1)-\psi(n_n+1)\right]+$$

$$+\frac{dn_p}{dt}\left[\psi(n_N+1)-\psi(n_p+1)\right]+$$

$$+\frac{dn_-}{dt}\left[\psi(n_\pi+1)-\psi(n_-+1)\right]+$$

$$+\frac{dn_0}{dt}\left[\psi(n_\pi+1)-\psi(n_0+1)\right]+$$

$$+\frac{dn_+}{dt}\left[\psi(n_\pi+1)-\psi(n_++1)\right]\ ;$$

$$n_N = n_n + n_p; \qquad n_\pi = n_- + n_0 + n_+$$