TOWARD CONVERGENCE of the variational EXPANSION for CHIRAL SYMMETRY BREAKING PARAMETERS

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related earlier work:
(Construction and applications of [this particular] "variational expansion"):

- Arvanitis, Geniet, Iacomi, JLK, Neveu '95-'96
  [mass-gap in O(N) Gross-Neveu model]
- Arvanitis, Geniet, JLK, Neveu '96
  JLK '96-'97
  [χSB order parameters in QCD]
  $\langle\bar{q}q\rangle$, F_π, $M_{dyn}(u,d,s)$

ICHEP 2000, Osaka, Japan, 27 Jul - 3 Aug. 2000

Introduction / Motivations

in QCD (or any AF theory with DYNAMICAL SB)
3 reasons (at least) why it is considered hopeless
to calculate the order parameters:
($\langle \bar{q} q \rangle \neq 0$, F_π, $M_q^{dyn} \gg m_q$, ..)
from "first principles" ($\mathcal{L}$ + basic pert. th.):

1. $\langle \bar{q} q \rangle^{1/3}$, F_π, ... $\sim \mathcal{O}(\Lambda_{QCD}) \simeq 100 - 300$ MeV
   $\to \alpha_s$ (relevant Q^2) large $\to$ invalidates pert. th.
2. $\langle \bar{q} q \rangle, .. \xrightarrow[m_q \to 0]{} 0$ at any pert. orders, trivial chiral limit
3. "impossible" to separate unambiguously NP/pert contributions
   (e.g. (Infrared) renormalon ambiguities)
   David '84 + many others

$\to$ "Best we can do" = OPE $= \sum c_q \langle O_q \rangle$

(SVZ (+ many others))   pert. coeff   NP parameters $\langle \bar{q} q \rangle$, $\langle G_{\mu\nu} G^{\mu\nu} \rangle$ ...

i.e. "soft versus hard"
[rather than pert/NP]

i.e: RIGOROUS ARGUMENTS (+ "conventional wisdom")
TELL US that (dynamical) χSB
= intrinsically NP ..

PLAN (SUMMARY)

reexamine a construction such that

I (2) $\langle \mathcal{O}_{\chi SB} \rangle \xrightarrow[m_q \to 0]{} 0$ circumvented

whereas (1) ($g^2(Q^2)$ large) also circumvented (irrelevant)

II All order generalization, however, spoils predictivity (infrared renormalon ambiguities)

III Combination (natural extension) with "variationally" improved δ-expansion

produces explicit damping of renormalon $n!$ divergences

→ Convergent (in some ren. scheme / approximation)

or Borel-summable (otherwise)

NB: Construction relevant to

$SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_V$ – χSB "order parameter" (QCD)

– mass gap of several 2D theories (Gross-Neveu model, etc) → γ_5 sym

I Dynamical χSB for $m \to 0$

- in a <u>generic</u> AFT, 1^{st} RG order

$$\beta(g) = \frac{dg}{d\ln\mu} = -b_0 g^3 + \dots, \; b_0 > 0 \qquad b_0^{QCD} = \frac{1}{16\pi^2}\left(11 - \tfrac{2}{3} n_f\right)$$

$$\gamma_m(g) = -\frac{d\ln(m)}{d\ln(\mu)} = \gamma_0 g^2 + \dots \qquad \gamma_0^{QCD} = \frac{1}{2\pi^2}$$

- neglect higher RG order + non-log corrections

SOLVE $\displaystyle m(\mu') = m(\mu) \exp\left[-\int_{g(\mu)}^{g(\mu')} dg \, \frac{\gamma_m(g)}{\beta(g)}\right]$

("running" mass)

with SELF-CONSISTANT "fixed point" condition:

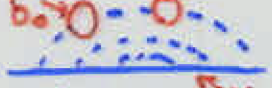
$m(\mu' = M) \equiv M$ ("lowest order" POLE mass)

$$\to \quad M = \underset{\substack{\uparrow \\ \text{running mass}}}{m(\mu)} \left[1 + 2b_0 \underset{\substack{\uparrow \\ \text{running coupling}}}{g^2(\mu)} \ln \frac{M}{\mu}\right]^{-\frac{\gamma_0}{2b_0}}$$

Note iteration

exhibits remarkable properties

- RG (scale) invariant to "all" orders

[

"all orders" BUT Leading Logs only!]

- gauge-inv (if gauge symmetry relevant)

- has non-trivial limit $M \to \Lambda$ for $m(\mu) \to 0$
uniquely selected (JLK '96)

More precisely, rewrite identically

$$M\left(\frac{\hat{m}}{\Lambda}\right) = \frac{\hat{m}}{\Lambda} F^{-A} \leftarrow \left(\begin{array}{l} \Lambda = \mu e^{-\frac{1}{2b_0 g^2(\mu)}} \equiv \text{"}\Lambda_{QCD}\text{" RG scale} \\ \hat{m} = m(\mu)[2b_0 g^2(\mu)]^{-A} \text{ scale-inv mass} \end{array} \right.$$

$$A = \frac{\gamma_0}{2b_0}$$

$$F = \frac{1}{2b_0 g^2(\mu)} = F\left(\frac{\hat{m}}{\Lambda}\right) = \ln \frac{\hat{m}}{\Lambda} - A \ln F$$

$$= A\, W\left(\frac{1}{A} \left(\frac{\hat{m}}{\Lambda}\right)^{1/A} \right) ; \quad W(x) \equiv \ln x - \ln W$$

Lambert fction or "ProductLog"

$$\cdot W(x) \sim \ln x - \ln(\ln x) \quad (x \to \infty)$$

But $$\underset{x \to 0}{=} x \left[1 + \sum_p (-1)^p \frac{(p+1)^p}{(p+1)!} x^p \right]$$

convergent for $|x| < 1/e$

$\rightarrow F\left(\frac{\hat{m}}{\Lambda}\right)$ provides a well-defined bridge

between the chiral sym. $\hat{m} \ll \Lambda$ (NP) regime

$$\left(\text{where } F \sim \left(\frac{\hat{m}}{\Lambda}\right)^{1/A} \left[1 + \mathcal{O}\left(\frac{\hat{m}}{\Lambda}\right)^{1/A} \right] \right.$$

power exp.

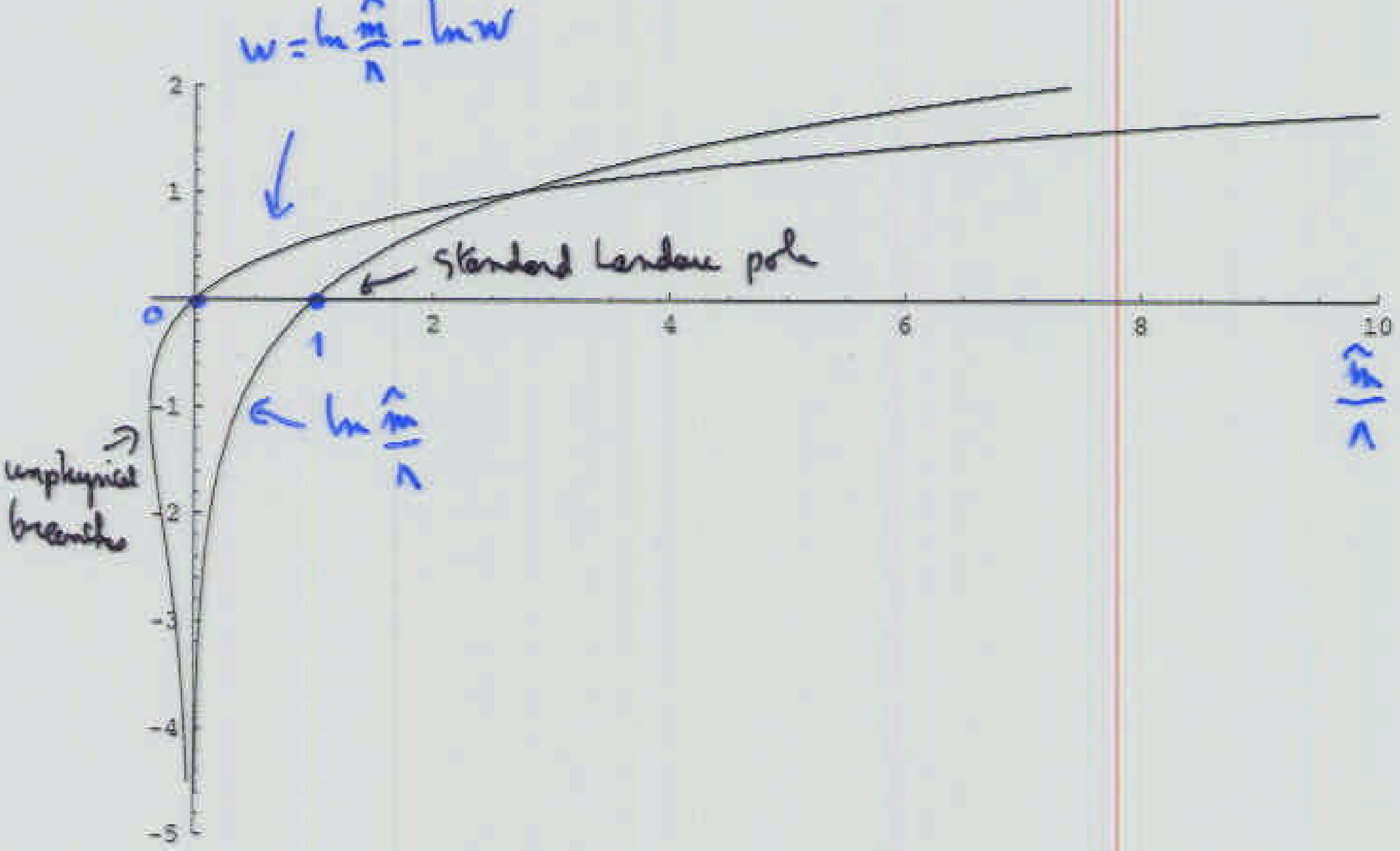
and the perturbative asympt. $\hat{m} \gg \Lambda$ regime

note F has singularities at $\frac{\hat{m}}{\Lambda} = (-1)^A e^{-A} A^A$

$\rightarrow$ Power expansion converges for $\left|\frac{\hat{m}}{\Lambda}\right| < e^{-A} A^A \sim \mathcal{O}(1)$

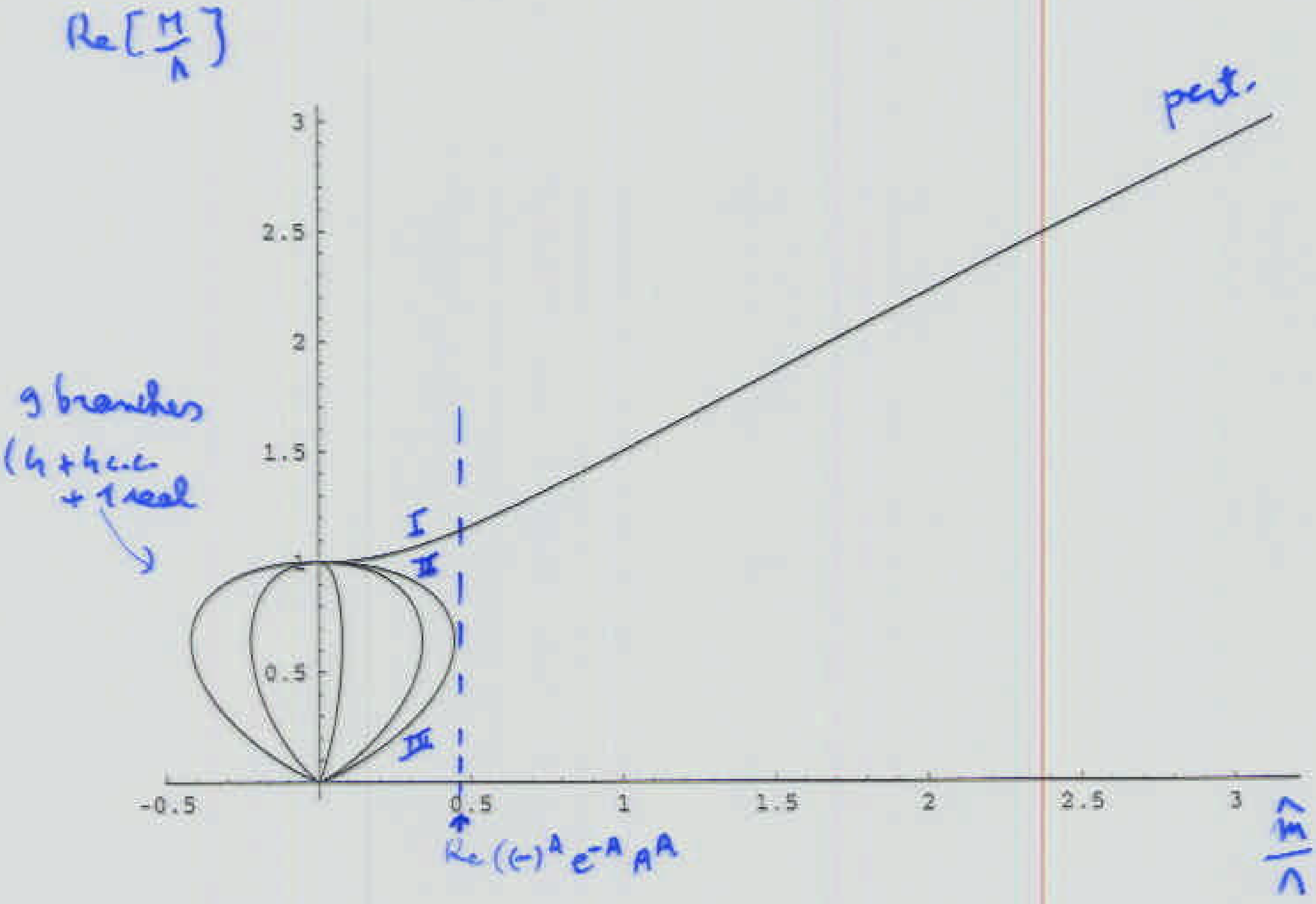
$$M \underset{\hat{m} \to 0}{\simeq} \frac{\hat{m}}{[(\hat{m}/\Lambda)^{1/A}]^A} \left(1 + \mathcal{O}\left(\frac{\hat{m}}{\Lambda}\right)^{1/A} \right)$$

$$= \Lambda \left[1 + \mathcal{O}\left(\frac{\hat{m}}{\Lambda}\right)^{1/A} \right]$$



$w = \ln \frac{\hat{m}}{\Lambda} - \ln w$
Standard Landau pole
$\ln \frac{\hat{m}}{\Lambda}$
unphysical branches
$\frac{\hat{m}}{\Lambda}$
0
1
2
4
6
8
10
2
1
-1
-2
-3
-4
-5

example with $A \equiv \frac{\gamma_0}{2b_0} \overset{(QCD)}{=} \frac{4}{9}$



- all branches are complex except I
- continuous matching with pert. behaviour

$\rightarrow \frac{M}{\Lambda} \underset{\hat{m}\to 0}{=} 1 + \mathcal{O}\left(\frac{\hat{m}}{\Lambda}\right)$ in I

• Higher orders – other χSB parameters

True POLE mass: $M^P = m(M^P)\left[1+\sum_{n\geq 1} c_n\, g^{2n}(M^P)\right]$

higher RG orders   non-log parts of  ;  , ...

Construction can be generalized

$$M^P(\hat m) = 2^{-C}\,\hat m\, F^{-A}(C+F)^{-B}\left[1+\sum_{n\geq 1}\frac{d_n}{(2b_0F)^n}\right]$$

$$F \equiv \ln\frac{\hat m}{\Lambda} - A\ln F - (B-C)\ln(C+F)$$ (exact 2-loop RG)

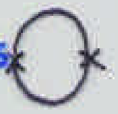
$A=\frac{\gamma_1}{2b_1}$; $B=\frac{\gamma_0}{2b_0}-\frac{\gamma_1}{2b_1}$, $C=\frac{b_1}{2b_0^2}$

with still $F(\hat m) \underset{\hat m\to 0}{\simeq} \left(\frac{\hat m}{\Lambda}\right)^{1/A}\left(1+O\left(\frac{\hat m}{\Lambda}\right)^{1/A}\right)$

Moreover, can be generalized also to χSB order parameters:

e.g. $i\langle 0|T\, J^{5i}_\mu J^{5j}_\nu|0\rangle = \delta^{ij} g_{\mu\nu} F_\pi^2 + O(p^2)$

$J^5_\mu = \bar q\gamma_\mu\gamma_5\frac{\lambda^i}{2}q$ $F_\pi\neq 0 \Rightarrow$ χSB

pert.th: $\lim_{\substack{p\to 0\\ m\neq 0}}$ $\gamma_\mu\gamma_5$  +  + ...

$$F_\pi^2 \sim 2^{-2C}\,\hat m^2\, F^{1-2A}(C+F)^{-2B}\,\delta_\pi\left[1+\sum_n\frac{d_n^{F_\pi}}{(2b_0F)^n}\right]$$

Same!

and $F_\pi \underset{\hat m\to 0}{\not\to} 0$!

• Similarly, for $m\langle\bar q q\rangle(\mu) \underset{(pert)}{=}$  +  + ...

$$\sim \hat m^4\, F^{1-4A}(C+F)^{-4B}\,\delta_{\bar qq}\left[1+\sum_n\frac{d_n^{\bar qq}}{(2b_0F)^n}\right]$$

GAVE correct order of magnitude estimates.. [Arvanitis, Geniet, JLK, Neveu '96
JLK '96–'97]

II Troubles with arbitrary higher orders

$$M^P \sim \hat{m}\, F^{-A} (c+F)^{-B} \left[1 + \sum_{n\geq 1} \frac{d_n}{(2b_0 F)^n} \right]$$

remarks: 1. non-log terms $\sim \frac{d_n}{(2b_0 F)^n} \underset{\hat{m}\to 0}{\sim} \left(\frac{\hat{m}}{\Lambda}\right)^{-n} \underset{\hat{m}\to 0}{\longrightarrow} \infty$

i.e. strict chiral limit undefined..

but not really a problem, e.g. in QCD, $m_q \neq 0$ natural IR cut-off
(also in GN IR cut-off $F_0 = -c > 0$)

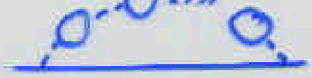
may even be convergent if $d_n \sim O(1)$ as $n \to \infty$

2. $A(\gamma_1)$, $B(\gamma_1)$, F, $d(n \geq 1)$ all are Renormalization Scheme dependent

(so that M^P remains RS invariant)

But, worst: 3. In fact, strong evidence that

$d_n \underset{n\to\infty}{\sim} n!$ (Infrared renormalons for AFT)

More precisely: $d_n \sim$  $\sim (2b_0)^n\, n!$

QCD: Beneke, Braun '94
(but similarly in GN at $1/N$ Reynaud, JLK)

$\rightarrow$ Ambiguities of $O(\Lambda)$ (from Borel transformation + Borel integration)

In our (special) case:

B.Int$(\hat{m}) \sim \hat{m}\, F^{-A} \left[1 + \int_0^\infty dt\, e^{-tF} \sum a^n \frac{n!}{n!} t^n \right.$ from B. transf.

$\rightarrow$ ambiguity $O(e^{-F/a}) \underset{\hat{m}\to 0}{\sim} O\left(e^{-\frac{1}{a}\left(\frac{\hat{m}}{\Lambda}\right)^{1/A}}\right)$

$\rightarrow$ predictibility lost..

III Natural Extension (cure): "variationally improved" Summation

Def: $\mathcal{L} \to \mathcal{L}_0(m=0) - m\bar{\Psi}\Psi + [x\, m\bar{\Psi}\Psi + \mathcal{L}_{int}(g \to g x)]$

$x=0$: $\mathcal{L}_{massive}$, free

$x=1$: $\mathcal{L}_{int}$ INDEPENDENT of m

$\to m \to m(1-x)$, $g \to g x$ at any pert. orders

(1^{st} order $\simeq$ Hartree-Fock)

idea : expand to finite order x^q $\to$ explicit m dependence

$\to m$ "variational" (adjustable) parameter

extrema $\frac{\partial}{\partial m}$ (physical quantities)$|_{x=1} \underset{\uparrow}{\simeq}$ Best approximation
hopefully

$\simeq$ Principle of Minimal Sensitivity (PMS) Stevenson '81

- Similar ideas developped in various forms

"δ-expansion" ($\delta \simeq x$) — Seznec, Zinn-Justin '79; Neveu '89

"order dependent mapping" — Bender et al '92; Bellet, Garcia, Neveu '95

also "Generalized Hartree-Fock" — H. Yamada '93 '94

MOREOVER, RIGOROUS CONVERGENCE PROOFS

(for several 1-D theories, an. oscillator)

most general proof → e.g. Guida, Konishi, Suzuki '95

put this method on more solid grounds,

but difficult to extend to $D \geq 2$ theories..

(Conv. proofs rely on "simple" analytic properties of 1-D theories)

However, in our (χSB) case: to keep | non-trivial $m\to 0$ properties
| consistency with renormalization

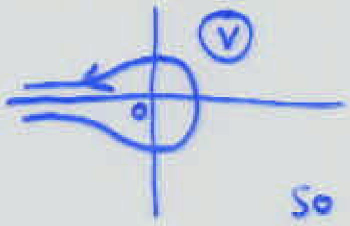
RESUM x^q series to "all" orders

(Bonus: expected already optimal % m)

$$\frac{M^P}{\Lambda}(\hat{m}') \underset{x\to 1}{\overset{q\to\infty}{=}} \frac{1}{2i\pi}\oint dv\; e^{v/\hat{m}'}\, F^{-A}(v)\,(C+F)^{-B}\left(1+\sum_{n\geq 1}\frac{d_n}{F^n}\right)$$

$x = 1-\frac{v}{q}$ $\qquad (1-\frac{v}{q})^{-(q+1)}\ (q\to\infty)$ $\qquad \hat{m}' = q\frac{\hat{m}}{\Lambda}$

where contour is: and $F(v) = \ln v - A\ln F - \ldots \underset{v\to 0}{\sim} v^{1/A}$



so $\frac{M^P}{\Lambda}(\hat{m}'\to 0) \sim \oint \frac{dv}{v}\, e^{v/\hat{m}'} \sum_n \frac{d_n}{v^{n/A}}$

$\sim$ const $\left[1+\sum \frac{d_n}{\Gamma(1+\frac{n}{A})}(\hat{m}')^{-\frac{n}{A}}\right]$ $\left(\frac{1}{2i\pi}\oint dv\, e^v v^\alpha = \frac{1}{\Gamma(-\alpha)}\right)$

NB $v\simeq 0$ dominates $\oint$ for $\hat{m}'\to 0$ (intuitive + rigorous (steepest contour) arguments)

Now $d_n \underset{n\to\infty}{\sim} (n-1)!\, a^n$ ($a = 1, 1/2, \ldots 1/k$ depending on quantity, or higher order sing.)

And $A = \frac{\gamma_1}{2b_1}$ RS-dependent!

$\to \sum \frac{(n-1)!}{\Gamma(1+\frac{n}{A})}(\hat{m}')^{-\frac{n}{A}}$ CONVERGENT for $0<A<1$
(divergent) (for $A>1$)
for all $\hat{m}>0$, all a $\qquad A^{QCD}_{\overline{MS}} \simeq 0.35\ (n_f=3)$

$\to$ IR renormalons "eaten up" provided $0<A<1$

[Actually, (slightly) more restrictive: $A = 1/k$, $k>0$ integer]

However $F(v) = v^{1/A}\left(1 + \sum (-)^p \frac{(p+1)^p}{(p+1)!} v^{p/A}\right)$

dominant for $\hat{m}' \to 0$ Contribute however

→ Full series (double $\sum_{n,p}$) more complicated..

original pert. F expansion

Results:
- for arbitrary finite p ($p/n \to 0$ as $n \to \infty$) still same convergence properties;
- for $p = \mathcal{O}(n)$ divergence reappears..
  BUT still Borel - summable
  (worst div. ~ $1/n$)

→ Asymptotic (full) series but no ambiguities (from renormalons)

(Final) remarks:

1. for $A > 1$, divergent (any p): "unphysical" sensitivity to RS?

   No, as remains Borel-Summable
   (i.e. smooth transition at $A = 1$)

2. $M(\hat{m}')_{A=1/K}$ explicitly summable as Hypergeometric $pF_q(\hat{m}'^{-1/A}) = \sum_n \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{(\hat{m}')^{-\frac{n}{A}}}{n!}$

$$(a)_n \equiv \frac{\Gamma(n+a)}{\Gamma(a)}$$

→ numerical analysis easy

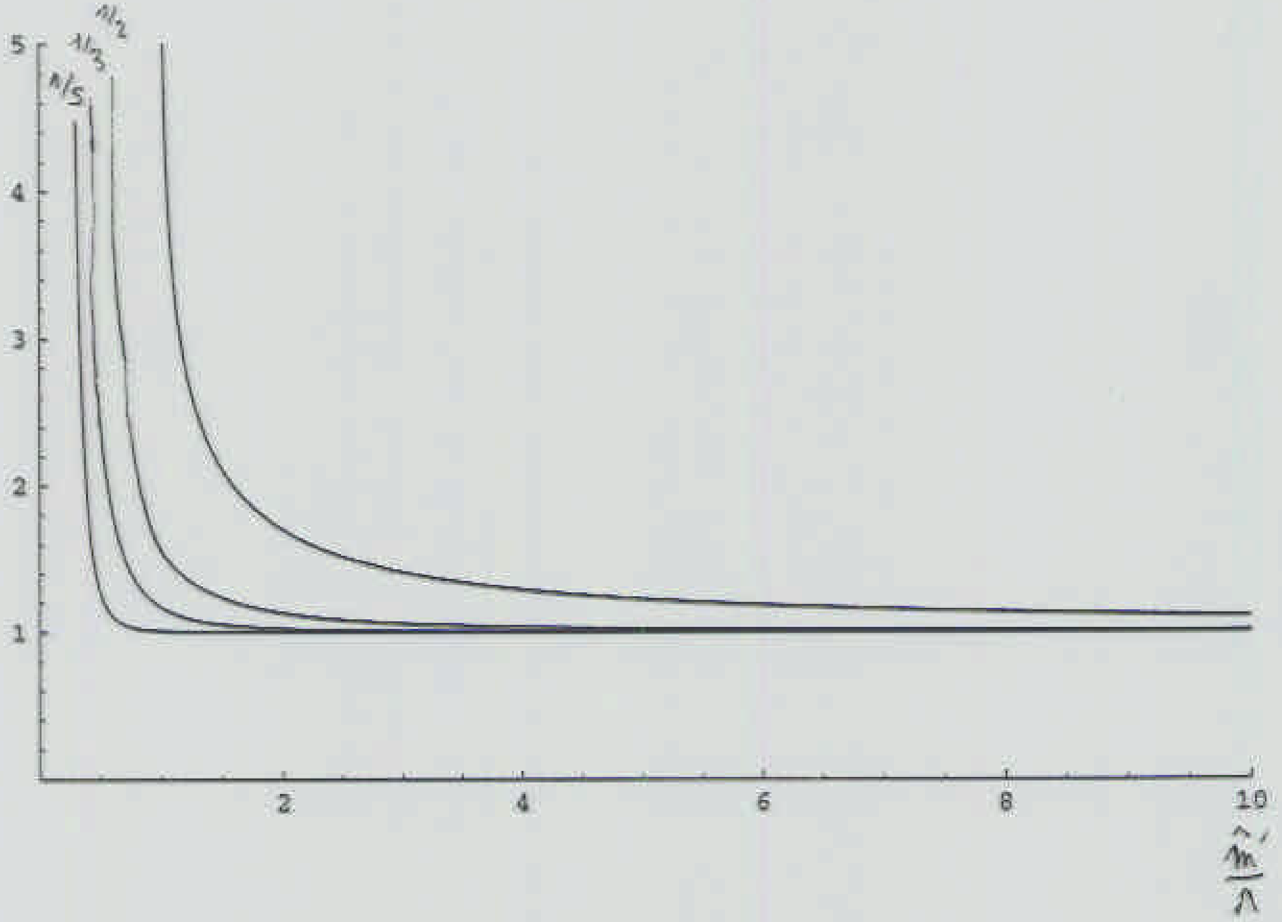
Moreover exhibits very flat dependence in $\hat{m}'$

Confirms x^q resummed ≃ no dependence in $\hat{m}$ (at least in some range)

Fig. →

$$1 + \sum \frac{(n-1)!}{\Gamma\left(1+\frac{n}{A}\right)} \left(m''\right)^{-\frac{n}{A}}$$

$A = 1$



SUMMARY / OUTLOOK

1. non trivial $m \to 0$ limit of χSB quantities, thanks to $F(\hat{m})$ properties for $\hat{m} \to 0$
(Counter example to conventional wisdom)

2. However, Consistent inclusion of arbitrary high perturbative terms $\to$ inherent ambiguities $O(\Lambda)$
($\simeq$ expected, but usually not seen in standard pert.)

3. "Variationally" improved x - summation produces factorial damping of pert. coeff.
$\to$ convergent (in leading $m \to 0$ approx.) or Borel summable (complete series)

$\to$ generalisation to higher dim. th of previously established conv. properties of 1-D theories?
(also conditionally dependent on adjustable parameters)

$\to$ New estimates of M_q^{dyn}, $\langle\bar{q}q\rangle$, F_π, etc under progress

* [Rk: is it surprising to get rid of IR renormalons?

not so much: normally, "exact" (NP) framework knows how to deal with renormalons
e.g in GN (ambiguities cancel with OPE part!)
Here, simply a way to do it "directly" due to nice properties of $F(\hat{m})$; $\hat{m} \to 0$
BUT! only relevant for χSB quantities
(that normally vanish for $m \to 0$)]