Quantum and Classical Gauge Symmetries

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1 Introduction

The Faddeev-Popov formula and the resulting BRST symmetry provide a basis for the modern quantization of gauge theory.
A modified quantization scheme

$$\begin{aligned}\int \mathcal{D}A_\mu\{\exp[-S_{YM}(A_\mu) - \int f(A_\mu)dx]\\ / \int \mathcal{D}g \exp[-\int f(A_\mu^g)dx]\}\end{aligned} \qquad (1.1)$$

with, for example,

$$f(A_\mu) = (m^2/2)(A_\mu)^2 \qquad (1.2)$$

has been recently analyzed in a **large mass** limit in connection with the analysis of Gribov-type complications. This gauge fixing in the large mass limit is equivalent to the non-linear gauge

$$A_\mu^2 = \lambda \qquad (1.3)$$

discussed by Dirac and Nambu many years ago.

The implementation of this non-linear gauge including the treatment of the apparent shift of the pole position has been discussd before. The above gauge fixing has also been used in lattice gauge theory.

We have recently shown that the above scheme is in fact identical at least in the perturbative accuracy to the conventional **local** Faddeev-Popov formula

$$\int \mathcal{D}A_\mu \quad \{\delta(D^\mu \frac{\delta f(A_\nu)}{\delta A_\mu}) / \int \mathcal{D}g \delta(D^\mu \frac{\delta f(A^g_\nu)}{\delta A^g_\mu})\} \\ \times \exp[-S_{YM}(A_\mu)]$$

$$= \int \mathcal{D}A_\mu \quad \delta(D^\mu \frac{\delta f(A_\nu)}{\delta A_\mu}) \det\{\delta[D^\mu \frac{\delta f(A^g_\nu)}{\delta A^g_\mu}]/\delta g\} \\ \times \exp[-S_{YM}(A_\mu)] \qquad (1.4)$$

without taking the large mass limit, if one takes into account the variation of the gauge field along the entire gauge orbit parametrized by the gauge parameter g. The above equivalence is valid only if the Gribov-type complications are ignored.

We here discuss the possible implications of the above equivalence in a more general context of quantum gauge symmetry, namely, BRST symmetry.

2 Abelian example

We first briefly illustrate the proof of the above equivalence of (1.1) and (1.4) by using an example of Abelian gauge theory,

$$S_0 = -(1/4) \int dx (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad (2.1)$$

for which we can work out everything explicitly.

In this note we exclusively work on Euclidean theory with metric convention $g_{\mu\nu} = (1,1,1,1)$.

As a simple and useful example, we choose the gauge fixing function $f(A) \equiv (1/2) A_\mu A_\mu$ and

$$D_\mu (\frac{\delta f}{\delta A_\mu}) = \partial_\mu A_\mu. \qquad (2.2)$$

Our claim above suggests the relation

$$\begin{aligned} Z = & \int \mathcal{D}A_\mu^\omega \{\exp\left[-S_0(A_\mu^\omega) - \int dx \frac{1}{2}(A_\mu^\omega)^2\right] \\ & / \int \mathcal{D}h \exp\left[-\int dx \frac{1}{2}(A_\mu^{h\omega})^2\right]\} \\ = & \int \mathcal{D}A_\mu^\omega \mathcal{D}B\mathcal{D}\bar{c}\mathcal{D}c \exp[-S_0(A_\mu^\omega) \\ & + \int(-iB\partial_\mu A_\mu^\omega + \bar{c}(-\partial_\mu\partial_\mu)c)dx] \end{aligned} \tag{2.3}$$

where the variable A_μ^ω stands for the field variable obtained from A_μ by a gauge transformation parametrized by the gauge orbit parameter ω.

To establish this result, we first evaluate

$$\begin{aligned} & \int \mathcal{D}h \exp[-\int dx \frac{1}{2}(A_\mu^{h\omega})^2] \\ & = \int \mathcal{D}h \exp[-\int dx \frac{1}{2}(A_\mu^\omega + \partial_\mu h)^2] \\ & = \int \mathcal{D}h \exp\{-\int dx \frac{1}{2}[(A_\mu^\omega)^2 - 2(\partial_\mu A_\mu^\omega)h + h(-\partial_\mu\partial_\mu)h]\} \\ & = \int \mathcal{D}B \frac{1}{det\sqrt{-\partial_\mu\partial_\mu}} \end{aligned}$$

$$\times \exp\{-\int dx \frac{1}{2}[(A_\mu^\omega)^2 - 2(\partial_\mu A_\mu^\omega)\frac{1}{\sqrt{-\partial_\mu\partial_\mu}}B + B^2]\}$$

$$= \frac{1}{det\sqrt{-\partial_\mu\partial_\mu}}$$

$$\times \exp[-\int dx \frac{1}{2}(A_\mu^\omega)^2 + \frac{1}{2}\int dx \partial_\mu A_\mu^\omega \frac{1}{-\partial_\mu\partial_\mu}\partial_\nu A_\nu^\omega] \tag{2.4}$$

where we defined $\sqrt{-\partial_\mu\partial_\mu}h = B$.

Thus

$$Z = \int \mathcal{D}A_\mu^\omega \{det\sqrt{-\partial_\mu\partial_\mu}\}$$

$$\times \exp\{-S_0(A_\mu^\omega) - \frac{1}{2}\int \partial_\mu A_\mu^\omega \frac{1}{-\partial_\mu\partial_\mu}\partial_\nu A_\nu^\omega dx\}$$

$$= \int \mathcal{D}A_\mu^\omega \mathcal{D}B\mathcal{D}\bar{c}\mathcal{D}c \exp\{-S_0(A_\mu^\omega) \tag{2.5}$$

$$-\frac{1}{2}\int B^2 dx + \int[-iB\frac{1}{\sqrt{-\partial_\mu\partial_\mu}}\partial_\mu A_\mu^\omega + \bar{c}\sqrt{-\partial_\mu\partial_\mu}c]dx\}$$

which is invariant under the BRST transformation

$$\delta A_\mu^\omega = i\lambda\partial_\mu c, \quad \delta c = 0$$

$$\delta\bar{c} = \lambda B, \quad \delta B = 0 \tag{2.6}$$

with a Grassmann parameter λ.

Note the appearance of the imaginary factor i in the term

$$iB\frac{1}{\sqrt{-\partial_\mu\partial_\mu}}\partial_\mu A^\omega_\mu \tag{2.7}$$

in (2.5).

We next rewrite the expression (2.5) as

$$\int \mathcal{D}A^\omega_\mu \mathcal{D}B\mathcal{D}\Lambda\mathcal{D}\bar{c}\mathcal{D}c\delta(\frac{1}{\sqrt{-\partial_\mu\partial_\mu}}\partial_\mu A^\omega_\mu - \Lambda)$$
$$\times \exp\{-S_0(A^\omega_\mu) - \frac{1}{2}\int(B^2 + 2i\Lambda B)dx + \int \bar{c}\sqrt{-\partial_\mu\partial_\mu}c dx\}$$
$$= \int \mathcal{D}A^\omega_\mu \mathcal{D}\Lambda\mathcal{D}\bar{c}\mathcal{D}c\delta(\frac{1}{\sqrt{-\partial_\mu\partial_\mu}}\partial_\mu A^\omega_\mu - \Lambda)$$
$$\times \exp\{-S_0(A^\omega_\mu) - \frac{1}{2}\int \Lambda^2 dx + \int \bar{c}\sqrt{-\partial_\mu\partial_\mu}c dx\}. \tag{2.8}$$

We note that we can compensate any variation of $\delta\Lambda$ by a suitable change of gauge parameter $\delta\omega$ inside the δ-function as

$$\frac{1}{\sqrt{-\partial_\mu \partial_\mu}} \partial_\mu \partial_\mu \delta\omega = \delta\Lambda. \qquad (2.9)$$

By a repeated application of infinitesimal gauge transformations combined with the invariance of the path integral measure under these gauge transformations, we can re-write the formula (2.8) as

$$\int \mathcal{D}A^\omega_\mu \mathcal{D}\Lambda \mathcal{D}\bar{c}\mathcal{D}c\,\delta(\frac{1}{\sqrt{-\partial_\mu \partial_\mu}} \partial_\mu A^\omega_\mu)$$
$$\times \exp\{-S_0(A^\omega_\mu) - \frac{1}{2}\int \Lambda^2 dx + \int \bar{c}\sqrt{-\partial_\mu \partial_\mu} c dx\}$$

$$= \int \mathcal{D}A^\omega_\mu \mathcal{D}\bar{c}\mathcal{D}c\,\delta(\frac{1}{\sqrt{-\partial_\mu \partial_\mu}} \partial_\mu A^\omega_\mu)$$
$$\times \exp\{-S_0(A^\omega_\mu) + \int \bar{c}\sqrt{-\partial_\mu \partial_\mu} c dx\}$$

$$= \int \mathcal{D}A^\omega_\mu \mathcal{D}B \mathcal{D}\bar{c}\mathcal{D}c$$
$$\times \exp\{-S_0(A^\omega_\mu) + \int [-iB \frac{1}{\sqrt{-\partial_\mu \partial_\mu}} \partial_\mu A^\omega_\mu + \bar{c}\sqrt{-\partial_\mu \partial_\mu} c] dx\}$$

$$= \int \mathcal{D}A^\omega_\mu \mathcal{D}B \mathcal{D}\bar{c}\mathcal{D}c$$
$$\times \exp\{-S_0(A^\omega_\mu) + \int [-iB \partial_\mu A^\omega_\mu + \bar{c}(-\partial_\mu \partial_\mu) c] dx\}. \qquad (2.10)$$

In the last stage of this equation, we re-defined the *auxiliary* variables B and $\bar{c}$ as

$$B \to B\sqrt{-\partial_\mu \partial_\mu}, \quad \bar{c} \to \bar{c}\sqrt{-\partial_\mu \partial_\mu} \tag{2.11}$$

which is consistent with BRST symmetry and leaves the path integral measure invariant. We have thus established the desired result (2.3).

It is shown that this procedure works for the non-Abelian case also, if the (ill-understood) Gribov-type complications can be ignored such as in perturbative calculations.

3 Possible Implications

In the classical level, we traditionally consider

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}m^2 A_\mu A^\mu \tag{3.1}$$

as a Lagrangian for a massive vector theory, and

$$\mathcal{L}_{eff} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{3.2}$$

as an effective Lagrangian for Maxwell theory with a Feynman-type gauge fixing term added. The physical meanings of these two Lagrangians are thus completely different.

However, the analysis in Section 2 shows that the Lagrangian (3.1) could in fact be regarded as a gauge fixed Lagrangian of *massless* Maxwell field in quantized theory.

To be explicit, by using (2.3), the Lagrangian (3.1) may be regarded as an effective Lagrangian in

$$\begin{aligned}
Z &= \int \mathcal{D}A_\mu^\omega \{\exp\{\int dx[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}m^2 A_\mu^\omega A^{\omega\mu}]\} \\
&\quad / \int \mathcal{D}h \exp[-\int dx \frac{m^2}{2}(A_\mu^{h\omega})^2]\} \\
&= \int \mathcal{D}A_\mu^\omega \mathcal{D}B\mathcal{D}\bar{c}\mathcal{D}c \exp\{\int dx[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\
&\qquad -iB\partial_\mu A_\mu^\omega + \bar{c}(-\partial_\mu\partial_\mu)c]\}. \qquad (3.3)
\end{aligned}$$

where we absorbed the factor m^2 into the definition of B and $\bar{c}$.

One can also analyze (3.2) by defining

$$f(A_\mu) \equiv \frac{1}{2}(\partial_\mu A^\mu)^2 \qquad (3.4)$$

in the modified quantization scheme (1.1).

The equality of (1.1) and (1.4) then gives

$$\int \mathcal{D}A_\mu \delta(D^\mu \frac{\delta f(A_\nu)}{\delta A_\mu}) \det\{\delta[D^\mu \frac{\delta f(A_\nu^g)}{\delta A_\mu^g}]/\delta g\} \exp[-S_0(A_\mu)]$$

$$= \int \mathcal{D}A_\mu \delta(\partial_\nu \partial^\nu (\partial^\mu A_\mu)) \det[\partial_\nu \partial^\nu \partial_\mu \partial^\mu] \exp[-S_0(A_\mu)]$$

$$= \int \mathcal{D}A_\mu \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \qquad (3.5)$$
$$\exp\{-S_0(A_\mu) + \int dx[-iB\partial_\nu \partial^\nu (\partial^\mu A_\mu) - \bar{c}(\partial_\nu \partial^\nu \partial_\mu \partial^\mu)c]\}$$

After the re-definition of *auxiliary* variables, $B\partial_\nu \partial^\nu \to B$, $\quad \bar{c}\partial_\nu \partial^\nu \to \bar{c}$, which preserves BRST symmetry, (3.5) becomes

$$\int \mathcal{D}A_\mu \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \exp\{-S_0(A_\mu) + \int dx[-iB(\partial^\mu A_\mu) + \bar{c}(-\partial_\mu \partial^\mu)c]\} \qquad (3.6)$$

which agrees with (2.8) and (3.3).

We can thus assign an identical physical meaning to two Lagrangians (3.1) and (3.2) in quantized theory.

Similarly, the two classical Lagrangians related to Yang-Mills fields

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 - \frac{m^2}{2} A_\mu^a A^{a\mu} \tag{3.7}$$

and

$$\mathcal{L}_{eff} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 - \frac{1}{2}(\partial_\mu A^{a\mu})^2 \tag{3.8}$$

could be assigned an identical physical meaning as an effective gauge fixed Lagrangian associated with the quantum theory defined by

$$\int \mathcal{D}A_\mu^a \mathcal{D}B^a \mathcal{D}\bar{c}^a \mathcal{D}c^a \exp\{-S_{YM}(A_\mu^a) + \int dx[-iB^a(\partial^\mu A_\mu^a) + \bar{c}^a(-\partial_\mu(D^\mu c)^a]\} \tag{3.9}$$

which is invariant under BRST symmetry.

We have illustrated that the apparent "massive gauge field" in the classical level has **no intrinsic** physical meaning.

It can be interpreted either as a classical massive (non-gauge) vector theory, or as a gauge-fixed effective Lagrangian for a massless gauge field.

In the framework of path integral, we have a certain freedom in the choice of the path integral measure:

One choice of the measure

$$\begin{aligned}
&\int d\mu \exp\{\int dx[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2 \\
&\qquad -\frac{m^2}{2} A^a_\mu A^{a\mu}]\} \\
&\equiv \int \mathcal{D}A_\mu \frac{1}{\int \mathcal{D}g \exp[-\int \frac{m^2}{2}(A^{ag}_\mu)^2 dx]\}} \\
&\times \exp\{\int dx[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2 \\
&\qquad -\frac{m^2}{2} A^a_\mu A^{a\mu}]\} \qquad (3.10)
\end{aligned}$$

gives rise to a massless gauge theory,

and the other choice

$$\int d\mu \exp\{\int dx[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2 - \frac{m^2}{2}A^a_\mu A^{a\mu}]\}$$
$$\equiv \int \mathcal{D}A_\mu \exp\{\int dx[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2 - \frac{m^2}{2}A^a_\mu A^{a\mu}]\} \quad (3.11)$$

gives rise to a massive *non-gauge* vector theory.

A somewhat analogous situation arises when one attempts to quantize the so-called anomalous gauge theory: A suitable choice of the measure with a Wess-Zumino term gives rise to a consistent quantum theory, if not renormalizable.

From a view point of classical-quantum correspondence, one can define a classical theory uniquely starting from quantum theory by considering the limit $\hbar \to 0$, but not the other way around in general.

In the context of the present interpretation of massive gauge fields, the massive gauge fields generated by the **Higgs mechanism** are exceptional and quite different.

Since all the terms including the mass term are gauge invariant, one can assign an intrinsic meaning to the massive gauge field in Higgs mechanism.

3.1 Possible origin of gauge fields

It is a long standing question if one can **generate gauge fields** from some *more* fundamental mechanism.

To our knowledge, however, there exists no definite convincing scheme so far. On the contrary, there is a no-go theorem or several arguments against such an attempt.

Apart from technical details, the basic argument against the "dynamical" generation of gauge fields is that the Lorentz invariant positive definite theory cannot simply generate the negative metric states associated with the time components of massless gauge fields.

In contrast, the **massive** "gauge fields" could be generated dynamically.

In general, the dynamical generation of the Lagrangian of the structure

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 - \frac{m^2}{2}(A_\mu^a)^2 \tag{3.12}$$

does not appear to be prohibited by general arguments so far.

If one considers that the induced Lagrangian such as (3.12) is a **classical** object which should be quantized anew, one could regard $\frac{m^2}{2}(A_\mu^a)^2$, which breaks classical gauge symmetry, as a **gauge fixing** term in the modified quantization scheme above.

In this interpretation, one might be allowed to say that massless gauge fields are generated dynamically. Although a dynamical generation of pure gauge fields is prohibited, a **gauge fixed** Lagrangian might be allowed to be generated.

We note that the mechanism for generating massless gauge fields by the violent random fluctuation of gauge degrees of freedom at the beginning of the universe is also closely related to the present observation.

An example of massive Abelian gauge field is analyzed in **compact** gauge theory defined on the lattice by Nielsen et al.. When one de-compactifies the theory one may be able to apply our analysis to their scheme also.

Incidentally, in the lattice simulation of QCD, the mass term as a gauge fixing term has also been used by Golterman et al..

References

[1] L.D. Faddeev and V.N. Popov, Phys. Lett. **B25**(1967)29.

[2] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. **42** (1975) 127; Ann. Phys. **98** (1976) 287.
J. Zinn-Justin, Lecture Notes in Physics, **37** (Springer-Verlag, Berlin, 1975)2.

[3] D. Zwanziger, Nucl.Phys.**B345**(1990)461, and references therein.

[4] G. Parrinello and G. Jona-Lasinio, Phys.Lett. **B251** (1990) 175.

[5] V.N. Gribov, Nucl. Phys. **B139**(1978)1.

[6] Y. Nambu, Suppl. Prog. Theor. Phys. Extra Number (1968)190.
P.A.M. Dirac, Proc. Roy. Soc.(London) **A209**(1951)291.

[7] K. Fujikawa, Phys. Rev. **D7**(1973)393.

[8] W. Bock, M. Golterman, M. Ogilvie and Y. Shamir, hep-lat/0004017.

[9] K. Fujikawa and H. Terashima, Nucl. Phys. **B577** (2000) 405.

[10] K. Fujikawa and H. Terashima, hep-th/0004190.

[11] K.M. Case and S. Gasiorowicz, Phys. Rev. **125**(1962)1055.
S. Coleman and E. Witten, Phys. Rev. Lett. **45**(1980)100.
S. Weinberg and E. Witten, Phys. Lett. **B96**(1980)59.

[12] D. Foerster, H.B. Nielsen and M. Ninomiya, Phys. Lett. **B94**(1980)135, and references therein.